
CHAPTER TEN

10.1 Derivability

1. a. Derive: $(\forall y)Fy$

1	$(\forall x)Fx$	Assumption
2	Fa	1 $\forall E$
3	$(\forall y)Fy$	2 $\forall I$

c. Derive: $(\exists x)(\exists y)Hxy$

1	$(\forall x)(\forall y)Hxy$	Assumption
2	$(\forall y)Hay$	1 $\forall E$
3	Hab	2 $\forall E$
4	$(\exists y)Hay$	3 $\exists I$
5	$(\exists x)(\exists y)Hxy$	4 $\exists I$

e. Derive: Kg

1	$(\forall x)(\forall y)Hxy$	Assumption
2	$Hab \supset Kg$	Assumption
3	$(\forall y)Hay$	1 $\forall E$
4	Hab	3 $\forall E$
5	Kg	2, 4 $\supset E$

g. Derive: $(\exists y)Wy$

1	$(\forall x)Sx$	Assumption
2	$(\exists y)Sy \supset (\forall w)Ww$	Assumption
3	Sa	1 $\forall E$
4	$(\exists y)Sy$	3 $\exists I$
5	$(\forall w)Ww$	2, 4 $\supset E$
6	Wa	5 $\forall E$
7	$(\exists y)Wy$	6 $\exists I$

i. Derive: $(\exists x)(Lxx \ \& \ Hxx)$

1	$(\forall x)(\forall y)Lxy$	Assumption		
2	$(\exists w)Hww$	Assumption		
3	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">Haa</td> <td style="padding-left: 20px;">A / $\exists E$</td> </tr> </table>	Haa	A / $\exists E$	
Haa	A / $\exists E$			
4	$(\forall y)Lay$	1 $\forall E$		
5	Laa	4 $\forall E$		
6	$Laa \ \& \ Haa$	3, 6 $\&I$		
7	$(\exists x)(Lxx \ \& \ Hxx)$	6 $\exists I$		
8	$(\exists x)(Lxx \ \& \ Hxx)$	2, 3–7 $\exists E$		

2. The mistakes in the attempted derivations are indicated and explained below.

a. Derive: Na

1	(∀x)Hx ⊃ ~ (∃y)Ky	Assumption	
2	Ha ⊃ Na	Assumption	
3	Ha	1 ∀E	MISTAKE!
4	Na	2, 3 ⊃E	

Universal Elimination is a rule of inference. Like all rules of inference, it can be applied only to whole sentences, not to a formula or sentence that is a component of a larger sentence, and ‘(∀x)Hx’ is a component of the larger sentence, namely ‘(∀x)Hx ⊃ ~ (∃y)Ky’.

c. Derive: (∃x)Cx

1	(∃y)Fy	Assumption	
2	(∀w)(Fw ≡ Cw)	Assumption	
3	Fa	1 ∃E	MISTAKE!
4	Fa ≡ Ca	2 ∀E	
5	Ca	3, 4 ≡E	
6	(∃x)Cx	5 ∃I	

Existential Elimination is a rule that requires the construction of a subderivation. Here is a correctly done derivation:

Derive: (∃x)Cx

1	(∃y)Fy	Assumption	
2	(∀w)(Fw ≡ Cw)	Assumption	
3	Fa	1 / ∃E	
4	Fa ≡ Ca	2 ∀E	
5	Ca	3, 4 ≡E	
6	(∃x)Cx	5 ∃I	
7	(∃x)Cx	2, 3–6 ∃E	

e. Derive: (∃y)(∀x)Ayx

1	(∀x)(∃y)Ayx	Assumption	
2	(∀x)Aax	1 ∀E	MISTAKE!
3	(∃y)(∀x)Ayx	2 ∃I	

Universal Elimination takes us from a Universally quantified sentence to a substitution instance of that sentence. Here we start with a universally quantified sentence but instead of dropping the universal quantifier the existential quantifier, which comes after the universal quantifier, has been dropped. There is no correct derivation in this case. The sentence on line 3 is not derivable in *PD* from the sentence on line 1.

10.2E EXERCISE ANSWERS

1. Validity

a. Derive: $(\forall x)(Fx \supset Hx)$

1		$(\forall y)[Fy \supset (Gy \ \& \ Hy)]$	Assumption
2		Fc	A / \supset I
3		$Fc \supset (Gc \ \& \ Hc)$	1 \forall E
4		$Gc \ \& \ Hc$	2, 3 \supset E
5		Hc	4 $\&$ E
6		$Fc \supset Hc$	2–5 \supset I
7		$(\forall x)(Fx \supset Hx)$	6 \forall I

#c. Our derivation of the conclusion from the premises will use Universal Elimination, Existential Elimination, and Existential Introduction. We will make Existential Elimination our primary strategy:

Derive: $(\exists z)Fz$

1		$(\forall y)[Gy \supset (Hy \ \& \ Fy)]$	Assumption
2		$(\exists x)Gx$	Assumption
3		Ga	A / \exists E
G		$(\exists z)Fz$	
G		$(\exists z)Fz$	2, 3— \exists E

We will next use Universal Elimination to obtain a material conditional whose antecedent is ‘Ga’, allowing us to use Conditional Elimination to obtain ‘Ha & Fa’. The rest is straightforward:

Derive: $(\exists z)Fz$

1		$(\forall y)[Gy \supset (Hy \ \& \ Fy)]$	Assumption
2		$(\exists x)Gx$	Assumption
3		Ga	A / \exists E
4		$Ga \supset (Ha \ \& \ Fa)$	1 \forall E
5		$Ha \ \& \ Fa$	3, 4 \supset E
6		Fa	5 $\&$ E
7		$(\exists z)Fz$	6 \exists I
8		$(\exists z)Fz$	2, 3–7 \exists E

e. Derive: $(\forall x)Hx$

1	$(\exists x)Fx \supset (\forall x)Gx$	Assumption
2	Fa	Assumption
3	$(\forall x)(Gx \supset Hx)$	Assumption
4	$(\exists x)Fx$	2 $\exists I$
5	$(\forall x)Gx$	1, 4 $\supset E$
6	Gb	5 $\forall E$
7	$Gb \supset Hb$	3 $\forall E$
8	Hb	6, 7 $\supset E$
9	$(\forall x)Hx$	8 $\forall I$

Note that it is essential that the constant chosen as the instantiating constant in line 6 be other than 'a', for 'a' occurs in an open assumption and were 'a' also used at line 6 we would violate the first restriction on Universal Introduction at line 9—for the instantiating constant, 'a', would then occur in an open assumption (on line 2).

g. Derive: $(\forall x)(Fx \vee Gx)$

1	$(\forall x)Fx \vee (\forall x)Gx$	Assumption
2	$(\forall x)Fx$	A / $\vee E$
3	Fa	2 $\forall E$
4	$Fa \vee Ga$	3 $\vee I$
5	$(\forall x)Gx$	A / $\vee E$
6	Ga	5 $\forall E$
7	$Fa \vee Ga$	6 $\vee I$
8	$Fa \vee Ga$	1, 2-4, 5-7 $\vee E$
9	$(\forall x)(Fx \vee Gx)$	8 $\forall I$

#i. Since the conclusion is a universally quantified sentence and there are no existentially quantified sentences among the premises, we will plan on deriving the conclusion by Universal Introduction and use Conditional Introduction to derive the substitution instance to which we will apply Universal Introduction:

Derive: $(\forall y)[(Fy \vee Gy) \supset Hy]$

1	$(\forall x)(Fx \supset Hx)$	Assumption
2	$(\forall y)(Gy \supset Hy)$	Assumption
3	$Fb \vee Gb$	A / $\supset I$
G	Hb	
G	$(Fb \vee Gb) \supset Hb$	3— $\supset I$
G	$(\forall y)[(Fy \vee Gy) \supset Hy]$	— $\forall I$

Our plan will not violate the second restriction on Universal Introduction, for while the instantiating constant 'b' does occur in an assumption (at line 3), that assumption will be closed at the point where we use Universal Introduction (the last line). The assumption on line 3 is a disjunction and we will now use Disjunction Elimination to obtain 'Hb'. To do so we will have to use Universal Elimination twice, once in association with each subderivation of the Disjunction Elimination strategy:

Derive: $(\forall y)[(Fy \vee Gy) \supset Hy]$

1	$(\forall x)(Fx \supset Hx)$	Assumption		
2	$(\forall y)(Gy \supset Hx)$	Assumption		
3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Fb \vee Gb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Fb \vee Gb		A / \supset I
Fb \vee Gb				
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Fb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Fb		A / \vee E
Fb				
5	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Fb \supset Hb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Fb \supset Hb		1 \forall E
Fb \supset Hb				
6	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Hb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Hb		4, 5 \supset E
Hb				
7	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Gb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Gb		A / \vee E
Gb				
8	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Gb \supset Hb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Gb \supset Hb		2 \forall E
Gb \supset Hb				
9	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Hb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Hb		7, 8 \supset E
Hb				
10	Hb	3, 4–6, 7–9 \vee E		
11	$(Fb \vee Gb) \supset Hb$	3–10 \supset I		
12	$(\forall y)[(Fy \vee Gy) \supset Hy]$	11 \forall I		

k. Derive: $(\forall x)(Fx \supset Gx)$

1	$(\exists x)Hx$	Assumption		
2	$(\forall x)(Hx \supset Rx)$	Assumption		
3	$(\exists x)Rx \supset (\forall x)Gx$	Assumption		
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Ha</td> <td style="padding-left: 5px;"></td> </tr> </table>	Ha		A / \exists E
Ha				
5	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Ha \supset Ra</td> <td style="padding-left: 5px;"></td> </tr> </table>	Ha \supset Ra		2 \forall E
Ha \supset Ra				
6	Ra	4, 5 \supset E		
7	$(\exists x)Rx$	6 \exists I		
8	$(\forall x)Gx$	3, 7 \supset E		
9	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Fb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Fb		A / \supset I
Fb				
10	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">Gb</td> <td style="padding-left: 5px;"></td> </tr> </table>	Gb		8 \forall E
Gb				
11	Fb \supset Gb	9–10 \supset I		
12	$(\forall x)(Fx \supset Gx)$	11 \forall I		
13	$(\forall x)(Fx \supset Gx)$	1, 4–12 \exists E		

m. Derive: $(\exists y)(Hy \vee Jy)$

1	$(\forall x)Fx \vee (\forall y) \sim Gy$		Assumption				
2	$Fa \supset Hb$		Assumption				
3	$\sim Gb \supset Jb$		Assumption				
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">4</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\forall x)Fx$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\forall E$</td> </tr> </table>	4	$(\forall x)Fx$		A / $\forall E$		A / $\forall E$
4	$(\forall x)Fx$		A / $\forall E$				
5	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">5</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">4 $\forall E$</td> </tr> </table>	5	Fa		4 $\forall E$		4 $\forall E$
5	Fa		4 $\forall E$				
6	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">6</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Hb </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">2, 5 $\supset E$</td> </tr> </table>	6	Hb		2, 5 $\supset E$		2, 5 $\supset E$
6	Hb		2, 5 $\supset E$				
7	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">7</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $Hb \vee Jb$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">6 $\vee I$</td> </tr> </table>	7	$Hb \vee Jb$		6 $\vee I$		6 $\vee I$
7	$Hb \vee Jb$		6 $\vee I$				
8	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">8</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\exists y)(Hy \vee Jy)$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">7 $\exists I$</td> </tr> </table>	8	$(\exists y)(Hy \vee Jy)$		7 $\exists I$		7 $\exists I$
8	$(\exists y)(Hy \vee Jy)$		7 $\exists I$				
9	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">9</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\forall y) \sim Gy$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\forall E$</td> </tr> </table>	9	$(\forall y) \sim Gy$		A / $\forall E$		A / $\forall E$
9	$(\forall y) \sim Gy$		A / $\forall E$				
10	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">10</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $\sim Gb$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">9 $\forall E$</td> </tr> </table>	10	$\sim Gb$		9 $\forall E$		9 $\forall E$
10	$\sim Gb$		9 $\forall E$				
11	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">11</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Jb </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">3, 10 $\supset E$</td> </tr> </table>	11	Jb		3, 10 $\supset E$		3, 10 $\supset E$
11	Jb		3, 10 $\supset E$				
12	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">12</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $Hb \vee Jb$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">11 $\vee I$</td> </tr> </table>	12	$Hb \vee Jb$		11 $\vee I$		11 $\vee I$
12	$Hb \vee Jb$		11 $\vee I$				
13	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">13</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\exists y)(Hy \vee Jy)$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">12 $\exists I$</td> </tr> </table>	13	$(\exists y)(Hy \vee Jy)$		12 $\exists I$		12 $\exists I$
13	$(\exists y)(Hy \vee Jy)$		12 $\exists I$				
14	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">14</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\exists y)(Hy \vee Jy)$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1, 4-8, 9-13 $\vee E$</td> </tr> </table>	14	$(\exists y)(Hy \vee Jy)$		1, 4-8, 9-13 $\vee E$		1, 4-8, 9-13 $\vee E$
14	$(\exists y)(Hy \vee Jy)$		1, 4-8, 9-13 $\vee E$				

2. Theorems

a. Derive: $Fa \supset (\exists y)Fy$

1	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">1</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\supset I$</td> </tr> </table>	1	Fa		A / $\supset I$		A / $\supset I$
1	Fa		A / $\supset I$				
2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">2</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\exists y)Fy$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1 $\exists I$</td> </tr> </table>	2	$(\exists y)Fy$		1 $\exists I$		1 $\exists I$
2	$(\exists y)Fy$		1 $\exists I$				
3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">3</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $Fa \supset (\exists y)Fy$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1-2 $\supset I$</td> </tr> </table>	3	$Fa \supset (\exists y)Fy$		1-2 $\supset I$		1-2 $\supset I$
3	$Fa \supset (\exists y)Fy$		1-2 $\supset I$				

c. Derive: $(\forall x)[Fx \supset (Gx \supset Fx)]$

1	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">1</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\supset I$</td> </tr> </table>	1	Fa		A / $\supset I$		A / $\supset I$				
1	Fa		A / $\supset I$								
2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">2</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> <table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">2</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Ga </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\supset I$</td> </tr> </table> </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\supset I$</td> </tr> </table>	2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">2</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Ga </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\supset I$</td> </tr> </table>	2	Ga		A / $\supset I$		A / $\supset I$		A / $\supset I$
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2	Ga		A / $\supset I$								
3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">3</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> <table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">3</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1 R</td> </tr> </table> </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1 R</td> </tr> </table>	3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">3</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1 R</td> </tr> </table>	3	Fa		1 R		1 R		1 R
3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">3</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1 R</td> </tr> </table>	3	Fa		1 R		1 R				
3	Fa		1 R								
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">4</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $Ga \supset Fa$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">2-3 $\supset I$</td> </tr> </table>	4	$Ga \supset Fa$		2-3 $\supset I$		2-3 $\supset I$				
4	$Ga \supset Fa$		2-3 $\supset I$								
5	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">5</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $Fa \supset (Ga \supset Fa)$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1-4 $\supset I$</td> </tr> </table>	5	$Fa \supset (Ga \supset Fa)$		1-4 $\supset I$		1-4 $\supset I$				
5	$Fa \supset (Ga \supset Fa)$		1-4 $\supset I$								
6	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">6</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\forall x)[Fx \supset (Gx \supset Fx)]$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">5 $\forall I$</td> </tr> </table>	6	$(\forall x)[Fx \supset (Gx \supset Fx)]$		5 $\forall I$		5 $\forall I$				
6	$(\forall x)[Fx \supset (Gx \supset Fx)]$		5 $\forall I$								

e. Derive: $\sim (\exists x)Fx \supset (\forall x) \sim Fx$

1	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">1</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $\sim (\exists x)Fx$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\supset I$</td> </tr> </table>	1	$\sim (\exists x)Fx$		A / $\supset I$		A / $\supset I$				
1	$\sim (\exists x)Fx$		A / $\supset I$								
2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">2</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> <table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">2</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\sim I$</td> </tr> </table> </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\sim I$</td> </tr> </table>	2	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">2</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> Fa </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">A / $\sim I$</td> </tr> </table>	2	Fa		A / $\sim I$		A / $\sim I$		A / $\sim I$
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3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">3</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\exists x)Fx$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">2 $\exists I$</td> </tr> </table>	3	$(\exists x)Fx$		2 $\exists I$		2 $\exists I$				
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4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">4</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $\sim (\exists x)Fx$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1 R</td> </tr> </table>	4	$\sim (\exists x)Fx$		1 R		1 R				
4	$\sim (\exists x)Fx$		1 R								
5	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">5</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $\sim Fa$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">2-4 $\sim I$</td> </tr> </table>	5	$\sim Fa$		2-4 $\sim I$		2-4 $\sim I$				
5	$\sim Fa$		2-4 $\sim I$								
6	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">6</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $(\forall x) \sim Fx$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">5 $\forall I$</td> </tr> </table>	6	$(\forall x) \sim Fx$		5 $\forall I$		5 $\forall I$				
6	$(\forall x) \sim Fx$		5 $\forall I$								
7	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">7</td> <td style="width: 15%; border-left: 1px solid black; padding-left: 5px;"> $\sim (\exists x)Fx \supset (\forall x) \sim Fx$ </td> <td style="width: 10%;"></td> <td style="width: 70%; vertical-align: top;">1-6 $\supset I$</td> </tr> </table>	7	$\sim (\exists x)Fx \supset (\forall x) \sim Fx$		1-6 $\supset I$		1-6 $\supset I$				
7	$\sim (\exists x)Fx \supset (\forall x) \sim Fx$		1-6 $\supset I$								

g. Derive: $Fa \vee (\exists y) \sim Fy$

1	$\sim (Fa \vee (\exists y) \sim Fy)$	$A / \sim E$
2	Fa	$A / \sim I$
3	$Fa \vee (\exists y) \sim Fy$	$2 \vee I$
4	$\sim (Fa \vee (\exists y) \sim Fy)$	$1 R$
5	$\sim Fa$	$2-4 \sim I$
6	$(\exists y) \sim Fy$	$5 \exists I$
7	$Fa \vee (\exists y) \sim Fy$	$6 \vee I$
8	$\sim (Fa \vee (\exists y) \sim Fy)$	$1 R$
9	$Fa \vee (\exists y) \sim Fy$	$1-8 \sim E$

#i. Since the theorem we want to prove is a material conditional, our primary strategy will be Conditional Introduction.

Derive: $[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$

1	$(\forall x)Fx \vee (\forall x)Gx$	$A / \supset I$
G	$(\forall x)(Fx \vee Gx)$	
G	$[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$	$1-_ \supset I$

Our only accessible assumption is a disjunction, and our current goal is a universally quantified sentence. This suggests we will be using both Disjunction Elimination and Universal Introduction. The question is whether the goal of our Disjunction Elimination strategy should be ' $(\forall x)(Fx \vee Gx)$ ' or a substitution instance of that sentence, say ' $Fb \vee Gb$ ', with the intent of using Universal Introduction after we have used Disjunction Elimination. It turns out that both approaches will work. We will use the latter approach:

Derive: $[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$

1	$(\forall x)Fx \vee (\forall x)Gx$	A / \supset I
2	$(\forall x)Fx$	A / \vee E
G	$Fb \vee Gb$	
	$(\forall x)Gx$	A / \vee E
G	$Fb \vee Gb$	
G	$Fb \vee Gb$	1, 2- \vee E, \vee E
G	$(\forall x)(Fx \vee Gx)$	\vee I
G	$[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$	1- \supset I

Completing the two Disjunction Elimination subderivations is straightforward. In each case we will use Universal Elimination followed by Disjunction Introduction. To make this work we must, of course, in both cases use ‘b’ as our instantiating constant:

Derive: $[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$

1	$(\forall x)Fx \vee (\forall x)Gx$	A / \supset I
2	$(\forall x)Fx$	A / \vee E
3	Fb	2 \forall E
4	$Fb \vee Gb$	3 \vee I
5	$(\forall x)Gx$	A / \vee E
6	Gb	5 \forall E
7	$Fb \vee Gb$	6 \vee I
8	$Fb \vee Gb$	1, 2-4, 5-7 \vee E
9	$(\forall x)(Fx \vee Gx)$	8 \forall I
10	$[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)$	1-9 \supset I

Note that we could have done Universal Introduction within each of our innermost subderivations, thereby obtaining ‘ $(\forall x)(Fx \vee Gx)$ ’ rather than ‘ $Fb \vee Gb$ ’ by Disjunction Elimination. Doing so would produce a derivation that is one line longer.

k. Derive: $(\exists x)(Fx \& Gx) \supset [(\exists x)Fx \& (\exists x)Gx]$

1	$(\exists x)(Fx \& Gx)$	A / \supset I
2	$Fa \& Ga$	A / \exists E
3	Fa	2 $\&$ E
4	$(\exists x)Fx$	3 \exists I
5	Ga	2 $\&$ E
6	$(\exists x)Gx$	5 \exists I
7	$(\exists x)Fx \& (\exists x)Gx$	4, 6 $\&$ I
8	$(\exists x)Fx \& (\exists x)Gx$	1, 2-7 \exists E
9	$(\exists x)(Fx \& Gx) \supset [(\exists x)Fx \& (\exists x)Gx]$	1-8 \supset I

m. Derive: $(\forall x)Hx \equiv \sim (\exists x) \sim Hx$

1		$(\forall x)Hx$	A / \equiv I
2		$(\exists x) \sim Hx$	A / \sim I
3		$\sim Ha$	A / \exists E
4		$(\forall x)Hx$	A / \sim I
5		$\sim Ha$	3 R
6		Ha	1 \forall E
7		$\sim (\forall x)Hx$	4-6 \sim I
8		$\sim (\forall x)Hx$	2, 3-7 \exists E
9		$(\forall x)Hx$	1 R
10		$\sim (\exists x) \sim Hx$	2-9 \sim I
11		$\sim (\exists x) \sim Hx$	A / \equiv I
12		$\sim Hb$	A / \sim E
13		$\sim (\exists x) \sim Hx$	11 R
14		$(\exists x) \sim Hx$	12 \exists I
15		Hb	12-14 \sim E
16		$(\forall x)Hx$	15 \forall I
17		$(\forall x)Hx \equiv \sim (\exists x) \sim Hx$	1-10, 11-16 \equiv I

3. Equivalence

a. Derive: $(\forall x)Fx \ \& \ (\forall x)Gx$

1		$(\forall x)(Fx \ \& \ Gx)$	Assumption
2		$Fa \ \& \ Ga$	1 \forall E
3		Fa	2 $\&$ E
4		$(\forall x)Fx$	3 \forall I
5		Ga	2 $\&$ E
6		$(\forall x)Gx$	5 \forall I
7		$(\forall x)Fx \ \& \ (\forall x)Gx$	4, 6 $\&$ I

Derive: $(\forall x)(Fx \ \& \ Gx)$

1		$(\forall x)Fx \ \& \ (\forall x)Gx$	Assumption
2		$(\forall x)Fx$	1 $\&$ E
3		Fa	2 \forall E
4		$(\forall x)Gx$	1 $\&$ E
5		Ga	4 \forall E
6		$Fa \ \& \ Ga$	3, 5 $\&$ I
7		$(\forall x)(Fx \ \& \ Gx)$	6 \forall I

c. Derive: $\sim (\exists x) \sim Fx$

1	$(\forall x)Fx$	Assumption
2	$(\exists x) \sim Fx$	A / \sim I
3	$\sim Fa$	A / \exists E
4	$(\forall x)Fx$	A / \sim I
5	Fa	4 \forall E
6	$\sim Fa$	3 R
7	$\sim (\forall x)Fx$	4-6 \sim I
9	$\sim (\forall x)Fx$	2, 3-7 \exists E
10	$(\forall x)Fx$	1 R
11	$\sim (\exists x) \sim Fx$	2-10 \sim I

Derive: $(\forall x)Fx$

1	$\sim (\exists x) \sim Fx$	Assumption
2	$\sim Fa$	A / \sim E
3	$(\exists x) \sim Fx$	2 \exists I
4	$\sim (\exists x) \sim Fx$	1 R
5	Fa	2-4 \sim E
6	$(\forall x)Fx$	5 \forall I

#e. Derive: $\sim (\forall x) \sim Fx$

1	$(\exists x)Fx$	Assumption
G	$\sim (\forall x) \sim Fx$	

The one primary assumption of our derivation is an existentially quantified sentence, suggesting Existential Elimination as a possible strategy. The goal sentence is a negation, suggesting Negation Introduction. In fact, we will use both strategies, one within the other. In our first attempt we will use Existential Elimination as our primary strategy:

Derive: $\sim (\forall x) \sim Fx$

1	$(\exists x)Fx$	Assumption
2	Fa	A / \exists E
3	$(\forall x) \sim Fx$	A / \sim I
G	$\sim (\forall x) \sim Fx$	3— \sim I
G	$\sim (\forall x) \sim Fx$	1, 2— \exists E

We have taken ' $\sim (\forall x) \sim Fx$ ' as our goal, within our Existential Elimination subderivation. Note that this sentence does not contain the constant 'a', so we are in no danger of violating the third restriction on Existential Elimination (that the instantiating constant not occur in the derived sentence). To complete the derivation we need to derive a sentence and its negation within the scope of the assumption on line 3. Only one negation is readily available, ' $\sim Fa$ ', which can be obtained by applying Universal Elimination to ' $(\forall x) \sim Fx$ ' on line 3. And ' Fa ' can be obtained by Reiteration. So the completed derivation is

Derive: $\sim (\forall x) \sim Fx$		
1	$(\exists x)Fx$	Assumption
2	Fa	A / $\exists E$
3	$(\forall x) \sim Fx$	A / $\sim I$
4	$\sim Fa$	3 $\forall E$
5	Fa	2 R
6	$\sim (\forall x) \sim Fx$	3-5 $\sim I$
7	$\sim (\forall x) \sim Fx$	1, 2-6 $\exists E$

To avoid violating the third restriction on Existential Elimination it is a good idea, at the time an Existential Elimination subderivation is started, to select the goal of that subderivation; making sure that the goal sentence does not contain the instantiating constant in the subderivation's assumption. In a derivation that uses Existential Elimination as its primary strategy the sentence that occurs on the last line should also appear as the last sentence in the subderivation. In this example that sentence is ' $\sim (\forall x) \sim Fx$ '.

To complete our demonstration that ' $(\exists x)Fx$ ' and ' $\sim (\forall x) \sim Fx$ ' are equivalent we will now derive the first sentence from the second:

Derive: $(\exists x)Fx$		
1	$\sim (\forall x) \sim Fx$	Assumption
G	$(\exists x)Fx$	

Here our goal sentence is an existentially quantified sentence, and our one primary assumption a negation. The former suggests Existential Introduction as a strategy, the latter suggests Negation Elimination (since we do have a negation readily available).

Derive: $(\exists x)Fx$		
1	$\sim (\forall x) \sim Fx$	Assumption
2	$\sim (\exists x)Fx$	A / \sim E
G	$(\forall x) \sim Fx$	
G	$\sim (\forall x) \sim Fx$	1 R
G	$(\exists x)Fx$	2- \sim E

We have decided to use ' $(\forall x) \sim Fx$ ' and ' $\sim (\forall x) \sim Fx$ ' as the sentence and negation we derive for Negation Elimination. (We could, of course, also have decided to use ' $(\exists x)Fx$ ' and ' $\sim (\exists x)Fx$ '.) Our current goal is ' $(\forall x) \sim Fx$ ', a universally quantified sentence. One way to obtain it is by Universal Introduction, which will require obtaining a substitution instance of that sentence. In planning for Universal Introduction we pick as our goal a substitution instance of the desired universally quantified sentence, and the instantiating constant in this substitution instance should not occur in any open assumption. Because neither of our assumptions contains a constant, we are free to choose any constant. We choose the substitution instance ' $\sim Fa$ '. And since this sentence is a negation, we will try to obtain it by Negation Introduction:

Derive: $(\exists x)Fx$		
1	$\sim (\forall x) \sim Fx$	Assumption
2	$\sim (\exists x)Fx$	A / \sim E
3	Fa	A / \sim I
G	$\sim Fa$	3- \sim I
G	$(\forall x) \sim Fx$	\sim \forall I
G	$\sim (\forall x) \sim Fx$	1 R
G	$(\exists x)Fx$	2- \sim E

As of line 3 two negations are available to us, ' $\sim (\forall x) \sim Fx$ ' and ' $\sim (\exists x)Fx$ '. We select the latter to use within the negation strategy that begins at line 3

because the unnegated ' $(\exists x)Fx$ ' is easily obtainable from line 3 by Existential Introduction:

Derive: $(\exists x)Fx$		
1	$\sim (\forall x) \sim Fx$	Assumption
2	$\sim (\exists x)Fx$	A / $\sim E$
3	Fa	A / $\sim I$
4	$(\exists x)Fx$	3 $\exists I$
5	$\sim (\exists x)Fx$	2 R
6	$\sim Fa$	3-5 $\sim I$
7	$(\forall x) \sim Fx$	6 $\forall I$
8	$\sim (\forall x) \sim Fx$	1 R
9	$(\exists x)Fx$	2-8 $\sim E$

We have now derived each member of our original pair of sentences from the other, so we have demonstrated that these sentences, ' $(\exists x)Fx$ ' and ' $\sim (\forall x) \sim Fx$ ' are equivalent in *PD*.

g. Derive: $\sim (\exists y)(Hy \ \& \ Iy)$

1	$(\forall z)(Hz \supset \sim Iz)$	Assumption
2	$(\exists y)(Hy \ \& \ Iy)$	A / $\sim I$
3	$Hb \ \& \ Ib$	A / $\exists E$
4	$(\forall z)(Hz \supset \sim Iz)$	A / $\sim I$
5	$Hb \supset \sim Ib$	1 $\forall E$
6	Hb	3 $\&E$
7	$\sim Ib$	5, 6 $\supset E$
8	Ib	3 $\&E$
9	$\sim (\forall z)(Hz \supset \sim Iz)$	4-8 $\sim I$
10	$\sim (\forall z)(Hz \supset \sim Iz)$	2, 3-9 $\exists E$
11	$(\forall z)(Hz \supset \sim Iz)$	1 R
12	$\sim (\exists y)(Hy \ \& \ Iy)$	2-11 $\sim I$

Derive: $(\forall z)(Hz \supset \sim Iz)$

1	$\sim (\exists y)(Hy \ \& \ Iy)$	Assumption
2	Ha	A / $\supset I$
3	Ia	A / $\sim I$
4	$Ha \ \& \ Ia$	2, 3 $\&I$
5	$(\exists y)(Hy \ \& \ Iy)$	4 $\exists I$
6	$\sim (\exists y)(Hy \ \& \ Iy)$	1 R
7	$\sim Ia$	3-6 $\sim I$
8	$Ha \supset \sim Ia$	2-7 $\supset I$
9	$(\forall z)(Hz \supset \sim Iz)$	8 $\forall I$

i. Derive: $(\forall x)(Fx \supset (\exists y)Gy)$

1		$(\forall x)(\exists y)(Fx \supset Gy)$	Assumption
2		$(\exists y)(Fa \supset Gy)$	1 $\forall E$
3		$Fa \supset Gb$	A / $\exists E$
4		Fa	A / $\supset I$
5		Gb	3, 4 $\supset E$
6		$(\exists y)Gy$	5 $\exists I$
7		$Fa \supset (\exists y)Gy$	4-6 $\supset I$
8		$Fa \supset (\exists y)Gy$	2, 3-7 $\exists E$
9		$(\forall x)(Fx \supset (\exists y)Gy)$	8 $\forall I$

Derive: $(\forall x)(\exists y)(Fx \supset Gy)$

1		$(\forall x)(Fx \supset (\exists y)Gy)$	Assumption
2		$\sim (\exists y)(Fa \supset Gy)$	A / $\sim E$
3		Fa	A / $\supset I$
4		$Fa \supset (\exists y)Gy$	1 $\forall E$
5		$(\exists y)Gy$	3, 4 $\supset E$
6		Gc	A / $\exists E$
7		$\sim Gb$	A / $\sim E$
8		Fa	A / $\supset I$
9		Gc	6 R
10		$Fa \supset Gc$	8-9 $\supset I$
11		$(\exists y)(Fa \supset Gy)$	10 $\exists I$
12		$\sim (\exists y)(Fa \supset Gy)$	2 R
13		Gb	7-12 $\sim E$
14		Gb	5, 6-13 $\exists E$
15		$Fa \supset Gb$	3-14 $\supset I$
16		$(\exists y)(Fa \supset Gy)$	15 $\exists I$
17		$\sim (\exists y)(Fa \supset Gy)$	2 R
18		$(\exists y)(Fa \supset Gy)$	2-17 $\sim E$
19		$(\forall x)(\exists y)(Fx \supset Gy)$	18 $\forall I$

4. Inconsistency

a. Derive: $Fa, \sim Fa$

1		$(\forall x)(Fx \equiv \sim Fx)$	Assumption
2		$Fa \equiv \sim Fa$	1 $\forall E$
3		Fa	A / $\sim I$
4		$\sim Fa$	2, 3 $\equiv E$
5		Fa	3 R
6		$\sim Fa$	3-5 $\sim I$
7		Fa	2, 6 $\equiv E$

#c. It is fairly easy to see that the set $\{\sim (\forall x)Fx, \sim (\exists x) \sim Fx\}$ is inconsistent. If not everything is F, then there must be something that is not F, but this contradicts the claim that there is not something that is not F. The set contains two negations. We choose to use one of them, ' $\sim (\forall x)Fx$ ', as $\sim Q$. Our derivation starts thus:

Derive: $(\forall x)Fx, \sim (\forall x)Fx$		
1	$\sim (\forall x)Fx$	Assumption
2	$\sim (\exists x) \sim Fx$	Assumption
G	$(\forall x)Fx$	
	$\sim (\forall x)Fx$	1 R

How we should continue is not immediately clear. We reason as follows: The sentences that are accessible include only two negations. There is no rule of inference that can be applied to a negation to yield a further sentence (Negation Elimination starts with the auxiliary assumption of a negation, not with a primary assumption that is a negation.) So working from the “top down” is not here promising. Our current goal is a universally quantified sentence, and Universal Introduction is the rule that yields such sentences. So we will plan on using Universal Introduction. To use it, we must first derive a substitution instance of our goal sentence. Since there are no constants in the primary assumptions, which substitution instance doesn't matter. We pick 'Fa'.

Derive: $(\forall x)Fx, \sim (\forall x)Fx$		
1	$\sim (\forall x)Fx$	Assumption
2	$\sim (\exists x) \sim Fx$	Assumption
G	Fa	
G	$(\forall x)Fx$	— $\forall I$
	$\sim (\forall x)Fx$	1 R

The task now is to derive 'Fa'. We have added no new assumptions, so working from the "top down" is still not promising. So we will try to get 'Fa' by Negation Elimination:

Derive: $(\forall x)Fx, \sim (\forall x)Fx$		
1	$\sim (\forall x)Fx$	Assumption
2	$\sim (\exists x) \sim Fx$	Assumption
3	$\sim Fa$	A / \sim E
	\vdots	
G	Fa	
G	$(\forall x)Fx$	$_ \forall I$
	$\sim (\forall x)Fx$	1 R

With our new assumption, we can now work from the "top down". More specifically, we have ' $\sim (\exists x) \sim Fx$ ' at line 2 and from line 3 we can obtain, by Existential Introduction, ' $(\exists x) \sim Fx$ ', giving us the \mathbf{Q} and $\sim \mathbf{Q}$ we need to complete our Negation Elimination strategy and the derivation:

Derive: $(\forall x)Fx, \sim (\forall x)Fx$		
1	$\sim (\forall x)Fx$	Assumption
2	$\sim (\exists x) \sim Fx$	Assumption
3	$\sim Fa$	A / \sim E
4	$(\exists x) \sim Fx$	3 $\exists I$
5	$\sim (\exists x) \sim Fx$	2 R
6	Fa	3-5 \sim E
7	$(\forall x)Fx$	6 $\forall I$
8	$\sim (\forall x)Fx$	1 R

Our demonstration of inconsistency in PD is now complete. We have used Universal Introduction and met both restrictions on that rule: the instantiating constant 'a' does not occur in the sentence derived by Universal Introduction and it does not occur, as of line 7, in any open assumption.

e. Derive: $(\exists x)Gx, \sim (\exists x)Gx$

1	$(\forall x)(Fx \supset Gx)$	Assumption
2	$(\exists x)Fx$	Assumption
3	$\sim (\exists x)Gx$	Assumption
4	Fb	A / $\exists E$
5	$Fb \supset Gb$	1 $\forall E$
6	Gb	4, 5 $\supset E$
7	$(\exists x)Gx$	6 $\exists I$
8	$(\exists x)Gx$	2, 4-7 $\exists E$
9	$\sim (\exists x)Gx$	3 R

g. Derive: $(\forall x)Fx, \sim (\forall x)Fx$

1	$(\forall x)Fx$	Assumption
2	$(\exists y) \sim Fy$	Assumption
3	$\sim Fa$	A / $\exists E$
4	$(\forall x)Fx$	A / $\sim I$
5	Fa	1 $\forall E$
6	$\sim Fa$	3 R
7	$\sim (\forall x)Fx$	4-6 $\sim I$
8	$\sim (\forall x)Fx$	2, 3-7 $\exists E$
9	$(\forall x)Fx$	1 R

i. Derive: $(\forall x)Fx, \sim (\forall x)Fx$

1	$(\forall x)(Hx \equiv \sim Gx)$	Assumption
2	$(\exists x)Hx$	Assumption
3	$(\forall x)Gx$	Assumption
4	Hc	A / $\exists E$
5	$(\forall x)Gx$	A / $\sim I$
6	$Hc \equiv \sim Gc$	1 $\forall E$
7	$\sim Gc$	4, 6 $\equiv E$
8	Gc	3 $\forall E$
9	$\sim (\forall x)Gx$	5-8 $\sim I$
10	$\sim (\forall x)Gx$	2, 4-9 $\exists E$
11	$(\forall x)Gx$	3 R

k. Derive: $(\exists y)(Ry \ \& \ My), \sim (\exists y)(Ry \ \& \ My)$

1	$(\forall z)[Rz \supset (Tz \ \& \ \sim Mz)]$	Assumption
2	$(\exists y)(Ry \ \& \ My)$	Assumption
3	$Ra \ \& \ Ma$	A / $\exists E$
4	$(\exists y)(Ry \ \& \ My)$	A / $\sim I$
5	$Ra \supset (Ta \ \& \ \sim Ma)$	1 $\forall E$
6	Ra	3 $\&E$
7	$Ta \ \& \ \sim Ma$	5, 6 $\supset E$
8	$\sim Ma$	7 $\&E$
9	Ma	3 $\&E$
10	$\sim (\exists y)(Ry \ \& \ My)$	4–9 $\sim I$
11	$\sim (\exists y)(Ry \ \& \ My)$	2, 3–10 $\exists E$
12	$(\exists y)(Ry \ \& \ My)$	2 R

5. Derivability

a. Derive: $(\forall x)(\exists y)Fxy$

1	$(\exists y)(\forall x)Fxy$	Assumption
2	$(\forall x)Fxa$	A / $\exists E$
3	Fba	2 $\forall E$
4	$(\exists y)Fby$	3 $\exists I$
5	$(\exists y)Fby$	1, 2–4 $\exists E$
6	$(\forall x)(\exists y)Fxy$	5 $\forall I$

c. Derive: $(\exists x)(\exists y)(\exists z)Fxyz$

1	$(\exists x)Fxxx$	Assumption
2	$Faaa$	A / $\exists E$
3	$(\exists z)Faaz$	2 $\exists I$
4	$(\exists y)(\exists z)Fayz$	3 $\exists I$
5	$(\exists x)(\exists y)(\exists z)Fxyz$	4 $\exists I$
6	$(\exists x)(\exists y)(\exists z)Fxyz$	1, 2–5 $\exists E$

e. Derive: $(\exists x)(\exists y)Gyx$

1	$(\forall x)(Fx \supset (\exists y)Gxy)$	Assumption
2	$(\exists x)Fx$	Assumption
3	Fa	A / $\exists E$
4	$Fa \supset (\exists y)Gay$	1 $\forall E$
5	$(\exists y)Gay$	3, 4 $\supset E$
6	Gab	A / $\exists E$
7	$(\exists y)Gyb$	6 $\exists I$
8	$(\exists x)(\exists y)Gyx$	7 $\exists I$
9	$(\exists x)(\exists y)Gyx$	5, 6–8 $\exists E$
10	$(\exists x)(\exists y)Gyx$	2, 3–9 $\exists E$

g. Derive: $(\exists x)(\exists y) \sim Hxy$

1	$(\forall x)(\forall y)(Hxy \supset \sim Hyx)$		Assumption
2	$(\exists x)(\exists y)Hxy$		Assumption
3	$(\exists y)Hxa$		A / $\exists E$
4	Hba		A / $\exists E$
5	$(\forall y)(Hby \supset \sim Hyb)$		1 $\forall E$
6	$Hba \supset \sim Hab$		5 $\forall E$
7	$\sim Hab$		4, 6 $\supset E$
8	$(\exists y) \sim Hyb$		7 $\exists I$
9	$(\exists x)(\exists y) \sim Hyx$		8 $\exists I$
10	$(\exists x)(\exists y) \sim Hxy$		3, 4–9 $\exists E$
11	$(\exists x)(\exists y) \sim Hxy$		2, 3–10 $\exists E$

i. Derive: $(\forall x)(\forall y)Hxy$

1	$\sim (\exists x)(\exists y)Rxy$		Assumption
2	$(\forall x)(\forall y)(\sim Hxy \equiv Rxy)$		Assumption
3	$\sim Hab$		A / $\sim E$
4	$(\forall y)(\sim Hay \equiv Ray)$		2 $\forall E$
5	$\sim Hab \equiv Rab$		4 $\forall E$
6	Rab		3, 5 $\equiv E$
7	$(\exists y)Ray$		6 $\exists I$
9	$(\exists x)(\exists y)Rxy$		7 $\exists I$
10	$\sim (\exists x)(\exists y)Rxy$		1 R
11	Hab		3–10 $\sim E$
12	$(\forall y)Hay$		11 $\forall I$
13	$(\forall x)(\forall y)Hxy$		12 $\forall I$

6. Validity

a. Derive: $(\exists y)Gya$

1	$(\forall x)(Fx \supset Gba)$		Assumption
2	$(\exists x)Fx$		Assumption
3	Fb		A / $\exists E$
4	$Fb \supset Gba$		1 $\forall E$
5	Gba		3, 4 $\supset E$
6	$(\exists y)Gya$		5 $\exists I$
7	$(\exists y)Gya$		2, 3–6 $\exists E$

c. Derive: $(\exists x)(\exists y)Fxy$

1	$(\exists x)(\exists y)(Fxy \vee Fyx)$		Assumption
2	$(\exists y)(Fay \vee Fya)$		A / $\exists E$
3	$Fab \vee Fba$		A / $\exists E$
4	Fab		A / $\vee E$
5	$(\exists y)Fay$		4 $\exists I$
6	$(\exists x)(\exists y)Fxy$		5 $\exists I$
7	Fba		A / $\vee E$
8	$(\exists y)Fby$		7 $\exists I$
9	$(\exists x)(\exists y)Fxy$		8 $\exists I$
10	$(\exists x)(\exists y)Fxy$		3, 4–6, 7–9 $\vee E$
11	$(\exists x)(\exists y)Fxy$		2, 3–10 $\exists E$
12	$(\exists x)(\exists y)Fxy$		1, 2–11 $\exists E$

e. Derive: $(\forall z)(Faz \supset Fza)$

1	$(\forall x)(\forall y)[(\exists z)[(Fyz \ \& \ \sim Fzx) \supset Gxy]$		Assumption
2	$\sim (\exists x)Gxx$		Assumption
3	Fab		A / $\supset I$
4	$\sim Fba$		A / $\sim E$
5	$(\forall y)[(\exists z)(Fyz \ \& \ \sim Fza) \supset Gay]$		1 $\forall E$
6	$(\exists z)(Faz \ \& \ \sim Fza) \supset Gaa$		5 $\forall E$
7	$Fab \ \& \ \sim Fba$		3, 4 $\&I$
8	$(\exists z)(Faz \ \& \ \sim Fza)$		7 $\exists I$
9	Gaa		6, 8 $\supset E$
10	$(\exists x)Gxx$		9 $\exists I$
11	$\sim (\exists x)Gxx$		2 R
12	Fba		4–11 $\sim E$
13	$Fab \supset Fba$		3–12 $\supset I$
14	$(\forall z)(Faz \supset Fza)$		13 $\forall I$

g. Derive: $(\forall x) \sim Fx$

1	$(\forall x)(Fx \supset (\exists y)Gxy)$		Assumption
2	$(\forall x)(\forall y) \sim Gxy$		Assumption
3	Fa		$A / \sim I$
4	$Fa \supset (\exists y)Gay$		1 $\forall E$
5	$(\exists y)Gay$		3, 4 $\supset E$
6	Gab		$A / \exists E$
7	$(\forall x)(\forall y) \sim Gxy$		$A / \sim I$
8	$(\forall y) \sim Gay$		2 $\forall E$
9	$\sim Gab$		8 $\forall E$
10	Gab		6 R
11	$\sim (\forall x)(\forall y) \sim Gxy$		7–11 $\sim I$
12	$\sim (\forall x)(\forall y) \sim Gxy$		5, 6–11 $\exists E$
13	$(\forall x)(\forall y) \sim Gxy$		2 R
14	$\sim Fa$		3–14 $\sim I$
15	$(\forall x)\sim Fx$		14 $\forall I$

7. Theorems

a. Derive: $(\forall x)(\exists z)(Fzx \supset Fzx)$

1	Faa		$A / \supset I$
2	Faa		1 R
3	$Faa \supset Faa$		1–2 $\supset I$
4	$(\exists z)(Faz \supset Fza)$		3 $\exists I$
5	$(\forall x)(\exists z)(Fzx \supset Fzx)$		4 $\forall I$

c. Derive: $(\forall x)(\forall y)Gxy \supset (\forall z)Gzz$

1	$(\forall x)(\forall y)Gxy$		$A / \supset I$
2	$(\forall y)Gay$		1 $\forall E$
3	Gaa		2 $\forall E$
4	$(\forall z)Gzz$		3 $\forall I$
5	$(\forall x)(\forall y)Gxy \supset (\forall z)Gzz$		1–4 $\supset I$

e. Derive: $(\forall x)Lxx \supset (\exists x)(\exists y)(Lxy \ \& \ Lyx)$

1	$(\forall x)Lxx$		$A / \supset I$
2	Laa		1 $\forall E$
3	$Laa \ \& \ Laa$		2, 2 $\&I$
4	$(\exists y)(Lay \ \& \ Lya)$		3 $\exists I$
5	$(\exists x)(\exists y)(Lxy \ \& \ Lyx)$		4 $\exists I$
6	$(\forall x)Lxx \supset (\exists x)(\exists y)(Lxy \ \& \ Lyx)$		1–5 $\supset I$

#g. The theorem to be proved, $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$ is a truth-functional compound whose main connective is a horseshoe. Therefore, we will use Conditional Introduction as our primary strategy:

Derive: $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$		
1	$(\exists x)(\forall y)Fxy$	Assumption
G	$(\exists x)(\exists y)Fxy$	
G	$(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$	$1\text{---}\supset I$

Our current goal is an existentially quantified sentence, $(\exists x)(\exists y)Fxy$. The most obvious way to obtain it is by two uses of Existential Introduction. Since the sentence on line 1 is an existentially quantified sentence it seems likely we will also be using Existential Elimination. And we know that when we do so, by assuming a substitution instance of $(\exists x)(\forall y)Fxy$, we will have to continue working within that subderivation until we obtain a sentence that does not contain the instantiating constant. This suggests that our current goal, $(\exists x)(\forall y)Fxy$, should also be the goal of our Existential Elimination subderivation, since it contains no constants:

Derive: $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$		
1	$(\exists x)(\forall y)Fxy$	Assumption
2	$(\forall y)Fay$	A / $\exists E$
G	$(\exists x)(\exists y)Fxy$	
G	$(\exists x)(\exists y)Fxy$	$2, 3\text{---}\exists E$
G	$(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$	$1\text{---}\supset I$

Completing this derivation is now straightforward. We use Universal Elimination on line 2 to produce 'Fab' and then use Existential Introduction twice to produce $(\exists x)(\exists y)Fxy$.

Derive: $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$

1	$(\exists x)(\forall y)Fxy$	Assumption
2	$(\forall y)Fay$	A / $\exists E$
3	Fab	2 $\forall E$
4	$(\exists y)Fay$	3 $\exists I$
5	$(\exists x)(\exists y)Fxy$	4 $\exists I$
6	$(\exists x)(\exists y)Fxy$	1, 2–5 $\exists E$
7	$(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$	1–6 $\supset I$

Here we do meet all the restrictions on Existential Elimination. The instantiating constant, which is here ‘a’, does not, at the point we use Existential Elimination (line 6) occur in any open assumption. The constant ‘a’ also does not occur in the existentially quantified sentence to which we are applying Existential Elimination, and it does not occur in the sentence derived by Existential Elimination (the sentence on line 6).

It is worth noting that since there are no restrictions on Existential Introduction, we could have entered, at line 3, ‘Faa’ rather than ‘Fab’ (there are also no restrictions on Universal Elimination), and then twice applied Existential Introduction.

i. Derive: $(\exists x)(\exists y)(Lxy \equiv Lyx)$

1	Laa	A / $\equiv I$
2	Laa	1 R
3	Laa \equiv Laa	1–2, 1–2 $\equiv I$
4	$(\exists y)(Lay \equiv Lya)$	3 $\exists I$
5	$(\exists x)(\exists y)(Lxy \equiv Lyx)$	4 $\exists I$

k. Derive: $(\forall x)(\forall y)(\forall z)Gxyz \supset (\forall x)(\forall y)(\forall z)(Gxyz \supset Gzyx)$

1	$(\forall x)(\forall y)(\forall z)Gxyz$	A / $\supset I$
2	$Gabc$	A / $\supset I$
3	$(\forall y)(\forall z)Gcyz$	1 $\forall E$
4	$(\forall z)Gcbz$	3 $\forall E$
5	Gcba	4 $\forall E$
6	Gabc \supset Gcba	2–5 $\supset I$
7	$(\forall z)(Gabz \supset Gzba)$	6 $\forall I$
8	$(\forall y)(\forall z)(Gayz \supset Gzya)$	7 $\forall I$
9	$(\forall x)(\forall y)(\forall z)(Gxyz \supset Gzyx)$	8 $\forall I$
10	$(\forall x)(\forall y)(\forall z)Gxyz \supset (\forall x)(\forall y)(\forall z)(Gxyz \supset Gzyx)$	1–9 $\supset I$

m. Derive: $(\forall x)(\forall y)(Fxy \equiv Fyx) \supset \sim (\exists x)(\exists y)(Fxy \& \sim Fyx)$

1	$(\forall x)(\forall y)(Fxy \equiv Fyx)$		A / \supset I
2	$(\exists x)(\exists y)(Fxy \& \sim Fyx)$		A / \sim I
3	$(\exists y)(Fay \& \sim Fya)$		A / \exists E
4	Fab & \sim Fba		A / \exists E
5	$(\forall x)(\forall y)(Fxy \equiv Fyx)$		A / \sim I
6	$(\forall y)(Fay \equiv Fya)$		1 \forall E
7	Fab \equiv Fba		6 \forall E
8	Fab		4 &E
9	Fba		7, 8 \equiv E
10	\sim Fba		4 &E
11	$\sim (\forall x)(\forall y)(Fxy \equiv Fyx)$		5–10 \sim I
12	$\sim (\forall x)(\forall y)(Fxy \equiv Fyx)$		3, 4–11 \exists E
13	$\sim (\forall x)(\forall y)(Fxy \equiv Fyx)$		2, 3–12 \exists E
14	$(\forall x)(\forall y)(Fxy \equiv Fyx)$		1 R
15	$\sim (\exists x)(\exists y)(Fxy \& \sim Fyx)$		2–14 \sim I
16	$(\forall x)(\forall y)(Fxy \equiv Fyx) \supset \sim (\exists x)(\exists y)(Fxy \& \sim Fyx)$		1–15 \supset I

8. Equivalence

a. Derive: $(\forall x)(Fx \supset (\exists y)Gya)$

1	$(\exists x)Fx \supset (\exists y)Gya$		Assumption
2	Fb		A / \supset I
3	$(\exists x)Fx$		2 \exists I
4	$(\exists y)Gya$		1, 3 \supset E
5	Fb $\supset (\exists y)Gya$		2–4 \supset I
6	$(\forall x)(Fx \supset (\exists y)Gya)$		5 \forall I

Derive: $(\exists x)Fx \supset (\exists y)Gya$

1	$(\forall x)(Fx \supset (\exists y)Gya)$		Assumption
2	$(\exists x)Fx$		A / \supset I
3	Fb		A / \exists E
4	Fb $\supset (\exists y)Gya$		1 \forall E
5	$(\exists y)Gya$		3, 4 \supset E
6	$(\exists y)Gya$		2, 3–5 \exists E
7	$(\exists x)Fx \supset (\exists y)Gya$		2–6 \supset I

#c. To establish that ' $(\exists x)[Fx \supset (\forall y)Hxy]$ ' and ' $(\exists x)(\forall y)(Fx \supset Hxy)$ ' are equivalent in *PD* we have to derive each from the unit set of the other. We begin by deriving ' $(\exists x)(\forall y)(Fx \supset Hxy)$ ' from $\{(\exists x)[Fx \supset (\forall y)Hxy]\}$. Since our one primary assumption will be an existentially quantified sentence we will

use Existential Elimination as our primary strategy and do virtually all of the derivation within that strategy:

Derive: $(\exists x)(\forall y)(Fx \supset Hxy)$		
1	$(\exists x)[Fx \supset (\forall y)Hxy]$	Assumption
2	$Fa \supset (\forall y)Hay$	A / $\exists E$
G	$(\exists x)(\forall y)(Fx \supset Hxy)$	
G	$(\exists x)(\forall y)(Fx \supset Hxy)$	1, 2— $\exists E$

Our current goal is an existentially quantified sentence. We will try to obtain it by Existential Introduction, and will try to obtain the required substitution instance, which will be a universally quantified sentence, by Universal Introduction:

Derive: $(\exists x)(\forall y)(Fx \supset Hxy)$		
1	$(\exists x)[Fx \supset (\forall y)Hxy]$	Assumption
2	$Fa \supset (\forall y)Hay$	A / $\exists E$
3		
G	$Fa \supset Hab$	
G	$(\forall y)(Fa \supset Hay)$	— $\forall I$
G	$(\exists x)(\forall y)(Fx \supset Hxy)$	— $\exists I$
G	$(\exists x)(\forall y)(Fx \supset Hxy)$	1, 2— $\exists E$

Our goal is now a material conditional, and we can obtain it by using Conditional Introduction and within that strategy Universal Elimination. The completed derivation is

Derive: $(\exists x)(\forall y)(Fx \supset Hxy)$

1	$(\exists x)[Fx \supset (\forall y)Hxy]$	Assumption
2	$Fa \supset (\forall y)Hay$	A / $\exists E$
3	Fa	A / $\supset I$
4	$(\forall y)Hay$	2, 3 $\supset E$
5	Hab	4 $\forall E$
6	$Fa \supset Hab$	3–5 $\supset I$
7	$(\forall y)(Fa \supset Hay)$	6 $\forall I$
8	$(\exists x)(\forall y)(Fx \supset Hxy)$	7 $\exists I$
9	$(\exists x)(\forall y)(Fx \supset Hxy)$	1, 2–8 $\exists E$

At line 5 we used Universal Elimination and in doing so were careful to pick a constant other than ‘a’ as our instantiating constant. Had we used ‘a’ we would not have been able to do Universal Introduction at line 7 because ‘a’ occurs in an assumption (the one on line 2) that is open as of line 7 and also occurs in line 7 itself.

When we apply Existential Elimination, at line 9, the instantiating constant, which is ‘a,’ does not occur in any open assumption, does not occur in the sentence we obtain at line 9, and of course does not occur in the existentially quantified sentence from which we are working (the sentence on line 1). So all three restrictions on Existential Elimination have been met. Note also that our use of Universal Introduction at line 7 meets both restrictions on that rule. The instantiating constant is ‘b’ and ‘b’ does not occur in any open assumption and does not occur in the sentence we obtain by Universal Introduction, ‘ $(\forall y)(Fa \supset Hay)$ ’

The derivation of ‘ $(\exists x)[Fx \supset (\forall y)Hxy]$ ’ from $\{(\exists x)(\forall y)(Fx \supset Hxy)\}$ is equally straightforward:

Derive: $(\exists x)[Fx \supset (\forall y)Hxy]$

1	$(\exists x)(\forall y)(Fx \supset Hxy)$	Assumption
2	$(\forall y)(Fa \supset Hay)$	A / $\exists E$
3	Fa	A / $\supset I$
4	$Fa \supset Hab$	2 $\forall E$
5	Hab	3, 4 $\supset E$
6	$(\forall y)Hay$	5 $\forall I$
7	$Fa \supset (\forall y)Hay$	3–6 $\supset I$
8	$(\exists x)[Fx \supset (\forall y)Hxy]$	7 $\exists I$
9	$(\exists x)[Fx \supset (\forall y)Hxy]$	1, 2–8 $\exists E$

We have again used Existential Elimination as our primary strategy and have again done the bulk of the work of the derivation within that strategy. We were again careful to pick an instantiating constant other than ‘a’ in doing Universal Elimination at line 4, again because using ‘a’ would prevent us from doing Universal Introduction at line 6.

e. Derive: $(\forall x)(\forall y)(Fxy \equiv \sim Gyx)$

1	$(\forall x)(\forall y) \sim (Fxy \equiv Gyx)$	Assumption
2	$(\forall y) \sim (Fay \equiv Gya)$	1 $\forall E$
3	$\sim (Fab \equiv Gba)$	2 $\forall E$
4	Fab	A / $\equiv I$
5	Gba	A / $\sim I$
6	Fab	A / $\equiv I$
7	Gba	5 R
8	Gba	A / $\equiv I$
9	Fab	4 R
10	Fab \equiv Gba	6-7, 8-9 $\equiv I$
11	$\sim (Fab \equiv Gba)$	3 R
12	\sim Gba	5-11 $\sim I$
13	\sim Gba	A / $\equiv I$
14	\sim Fab	A / $\sim E$
15	Fab	A / $\equiv I$
16	\sim Gba	A / $\sim I$
17	Fab	15 R
18	\sim Fab	14 R
19	Gba	16-18 $\sim E$
20	Gba	A / $\equiv I$
21	\sim Fab	A / $\sim E$
22	Gba	20 R
23	\sim Gba	13 R
24	Fab	21-23 $\sim E$
25	Fab \equiv Gba	15-19, 20-24 $\equiv I$
26	$\sim (Fab \equiv Gba)$	3 R
27	Fab	14-26 $\sim E$
28	Fab $\equiv \sim$ Gba	4-12, 13-27 $\equiv I$
29	$(\forall y)(Fay \equiv \sim Gya)$	28 $\forall I$
30	$(\forall x)(\forall y)(Fxy \equiv \sim Gyx)$	29 $\forall I$

Derive: $(\forall x)(\forall y) \sim (Fxy \equiv Gyx)$

1	$(\forall x)(\forall y)(Fxy \equiv \sim Gyx)$		Assumption
2	$Fab \equiv Gba$		A / \sim I
3	$(\forall y)(Fay \equiv \sim Gya)$		1 \forall E
4	$Fab \equiv \sim Gba$		3 \forall E
5	Fab		A \sim I
6	$\sim Gba$		4, 5 \equiv E
7	Gba		2, 5 \equiv E
8	$\sim Fab$		5-7 \sim I
9	$\sim Gba$		A / \sim E
10	Fab		4, 9 \equiv E
11	Gba		2, 10 \equiv E
12	$\sim Gba$		9 R
13	Gba		9-12 \sim E
14	Fab		2, 13 \equiv E
15	$\sim (Fab \equiv Gba)$		2-14 \sim I
16	$(\forall y) \sim (Fay \equiv Gya)$		15 \forall I
17	$(\forall x)(\forall y) \sim (Fxy \equiv Gyx)$		16 \forall I

9. Inconsistency

a. Derive: $Tab, \sim Tab$

1	$(\forall x)(\forall y)[(Ex \ \& \ Ey) \supset Txy]$		Assumption
2	$(Ea \ \& \ Eb) \ \& \ \sim Tab$		Assumption
3	$(\forall y)[(Ea \ \& \ Ey) \supset Tay]$		1 \forall E
4	$(Ea \ \& \ Eb) \supset Tab$		3 \forall E
5	$Ea \ \& \ Eb$		2 $\&$ E
6	Tab		4, 5 \supset E
7	$\sim Tab$		2 $\&$ E

c. Derive: $(\exists x)Fxx, \sim (\exists x)Fxx$

1	$\sim (\exists x)Fxx$		Assumption
2	$(\exists x)(\forall y)Fxy$		Assumption
3	$(\forall y)Fay$		A / \exists E
4	Faa		3 \forall E
5	$(\exists x)Fxx$		4 \exists I
6	$(\exists x)Fxx$		2, 3-5 \exists E
7	$\sim (\exists x)Fxx$		1 R

e. Derive: $(\forall y) \sim Lay, \sim (\forall y) \sim Lay$

1	$(\forall x)(\exists y)Lxy$	Assumption
2	$(\forall y) \sim Lay$	Assumption
3	$(\exists y)Lay$	1 $\forall E$
4	Lab	A / $\exists E$
5	$(\forall y) \sim Lay$	A / $\sim I$
6	$\sim Lab$	5 $\forall E$
7	Lab	4 R
8	$\sim (\forall y) \sim Lay$	5-7 $\sim I$
9	$\sim (\forall y) \sim Lay$	3, 4-8 $\exists E$
10	$(\forall y) \sim Lay$	2 R

g. Derive: $(\exists x) \sim (\exists y)Lyx, \sim (\exists x) \sim (\exists y)Lyx$

1	$(\forall x)[Hx \supset (\exists y)Lyx]$	Assumption
2	$(\exists x) \sim (\exists y)Lyx$	Assumption
3	$(\forall x)Hx$	Assumption
4	$\sim (\exists y)Lya$	A / $\exists E$
5	$(\exists x) \sim (\exists y)Lyx$	A / $\sim I$
5	$Ha \supset (\exists y)Lya$	1 $\forall E$
6	Ha	3 $\forall E$
7	$(\exists y)Lya$	5, 6 $\supset E$
8	$\sim (\exists y)Lya$	4 R
9	$\sim (\exists x) \sim (\exists y)Lyx$	5-8 $\sim I$
10	$\sim (\exists x) \sim (\exists y)Lyx$	2, 4-9 $\exists E$
11	$(\exists x) \sim (\exists y)Lyx$	2 R

#i. We will now show that the set $\{(\forall x)(\exists y)Fxy, (\exists z) \sim (\exists w)Fzw\}$ is inconsistent in *PD*. This is an interesting problem in several respects. Neither set member is a negation. So it is not obvious which pair of contradictory sentences (the **Q** and $\sim \mathbf{Q}$ we must derive to show the set is contradictory) we should take as our goal. One of the set members is an existentially quantified sentence, so it is plausible that our derivation will involve an Existential Elimination as its main strategy, with a substitution instance of ' $(\exists z) \sim (\exists w)Fzw$ ' as the assumption of a subderivation. Remembering that it is often useful to do as much of the work of a derivation as possible within an Existential Elimination subderivation we will make Existential Elimination our primary strategy:

Derive: ?, ?

1	$(\forall x)(\exists y)Fxy$	Assumption
2	$(\exists z) \sim (\exists w)Fzw$	Assumption
3	$\sim (\exists w)Faw$	A / $\exists E$

Our new assumption is a negation, but there is obviously no hope of moving that sentence out from within the scope of our subderivation so that it can play the role of $\sim Q$ in our derivation – no hope because it obviously contains the instantiating constant ‘a’. A better strategy is to try to obtain a negation within the scope of the Existential Elimination strategy that does not contain the constant ‘a’. The obviously useful negation is ‘ $\sim (\forall x)(\exists y)Fxy$ ’ because we can obtain the sentence of which it is the negation, ‘ $(\forall x)(\exists y)Fxy$ ’ by Reiteration on line 1. So we will proceed as follows:

Derive: $(\forall x)(\exists y)Fxy, \sim (\forall x)(\exists y)Fxy$

1	$(\forall x)(\exists y)Fxy$	Assumption
2	$(\exists z) \sim (\exists w)Fzw$	Assumption
3	<div style="border-left: 1px solid black; padding-left: 10px;">$\sim (\exists w)Faw$</div>	A / $\exists E$
4	<div style="border-left: 1px solid black; padding-left: 10px;">$(\forall x)(\exists y)Fxy$</div>	A / $\sim I$
G	$\sim (\forall x)(\exists y)Fxy$	$_ _ _ \sim I$
G	$\sim (\forall x)(\exists y)Fxy$	2, 3- $_ _ _ \exists E$
	$(\forall x)(\exists y)Fxy$	1 R

We now need to derive a sentence and its negation within the scope of the assumption on line 4. There is no reason not to use the negation on line 3. We will do so, making our new goal ‘ $(\exists w)Faw$ ’:

Derive: $(\forall x)(\exists y)Fxy, \sim (\forall x)(\exists y)Fxy$

1	$(\forall x)(\exists y)Fxy$	Assumption
2	$(\exists z) \sim (\exists w)Fzw$	Assumption
3	<div style="border-left: 1px solid black; padding-left: 10px;">$\sim (\exists w)Faw$</div>	A / $\exists E$
4	<div style="border-left: 1px solid black; padding-left: 10px;">$(\forall x)(\exists y)Fxy$</div>	A / $\sim I$
G	<div style="border-left: 1px solid black; padding-left: 10px;">$(\exists w)Faw$</div>	3 R
	$\sim (\exists w)Faw$	$_ _ _ \sim I$
G	$\sim (\forall x)(\exists y)Fxy$	2, 3- $_ _ _ \exists E$
G	$\sim (\forall x)(\exists y)Fxy$	1 R

From line 1 we can obtain ‘ $(\exists y)Fay$ ’ by Universal Elimination. And we can move from ‘ $(\exists y)Fay$ ’ to ‘ $(\exists w)Faw$ ’ by an Existential Elimination strategy. Our completed derivation is

Derive: $(\forall x)(\exists y)Fxy, \sim (\forall x)(\exists y)Fxy$

1	$(\forall x)(\exists y)Fxy$	Assumption
2	$(\exists z) \sim (\exists w)Fzw$	Assumption
3	<div style="border-left: 1px solid black; padding-left: 10px;">$\sim (\exists w)Faw$</div>	$A / \exists E$
4	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-left: 1px solid black; padding-left: 10px;">$(\forall x)(\exists y)Fxy$</div> </div>	$A / \sim I$
5	<div style="border-left: 1px solid black; padding-left: 10px;">$(\exists y)Fay$</div>	1 $\forall E$
6	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-left: 1px solid black; padding-left: 10px;">Fab</div> </div>	$A / \exists E$
7	<div style="border-left: 1px solid black; padding-left: 10px;">$(\exists w)Faw$</div>	6 $\exists I$
8	$(\exists w)Faw$	5, 6–7 $\exists E$
9	$\sim (\exists w)Faw$	3 R
10	$\sim (\forall x)(\exists y)Fxy$	4–9 $\sim I$
11	$\sim (\forall x)(\exists y)Fxy$	2, 3–10 $\exists E$
12	$(\forall x)(\exists y)Fxy$	1 R

We have used Existential Elimination twice and in both instances we met all restrictions on that rule. In the first use, at line 8, the instantiating constant is ‘b’ and ‘b’ does not occur in either line 5 or line 8 and it does not, as of line 8, occur in any open assumption.

k. Derive: $(\forall x)(\forall y)(Fxy \vee Gxy), \sim (\forall x)(\forall y)(Fxy \vee Gxy)$

1	$(\forall x)(\forall y)(Fxy \vee Gxy)$	Assumption
2	$(\exists x)(\exists y)(\sim Fxy \ \& \ \sim Gxy)$	Assumption
3	<div style="border-left: 1px solid black; padding-left: 10px;">$(\exists y)(\sim Fay \ \& \ \sim Gay)$</div>	$A / \exists E$
4	<div style="border-left: 1px solid black; padding-left: 10px;">$\sim Fab \ \& \ \sim Gab$</div>	$A / \exists E$
5	<div style="border-left: 1px solid black; padding-left: 10px;">$(\forall y)(Fay \vee Gay)$</div>	1 $\forall E$
6	$Fab \vee Gab$	5 $\forall E$
7	<div style="border-left: 1px solid black; padding-left: 10px;">Fab</div>	$A / \vee E$
8	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-left: 1px solid black; padding-left: 10px;">$(\forall x)(\forall y)(Fxy \vee Gxy)$</div> </div>	$A / \sim I$
9	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-left: 1px solid black; padding-left: 10px;">Fab</div> </div>	7 R
10	<div style="border-left: 1px solid black; padding-left: 10px;">$\sim Fab$</div>	4 $\&E$
11	$\sim (\forall x)(\forall y)(Fxy \vee Gxy)$	8–10 $\sim I$
12	<div style="border-left: 1px solid black; padding-left: 10px;">Gab</div>	$A \vee E$
13	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-left: 1px solid black; padding-left: 10px;">$(\forall x)(\forall y)(Fxy \vee Gxy)$</div> </div>	$A / \sim I$
14	<div style="border-left: 1px solid black; padding-left: 10px;"> <div style="border-left: 1px solid black; padding-left: 10px;">Gab</div> </div>	14 R
15	<div style="border-left: 1px solid black; padding-left: 10px;">$\sim Gab$</div>	4 $\&E$
16	$\sim (\forall x)(\forall y)(Fxy \vee Gxy)$	13–15 $\sim I$
17	$\sim (\forall x)(\forall y)(Fxy \vee Gxy)$	6, 7–11, 12–16 $\vee E$
18	$\sim (\forall x)(\forall y)(Fxy \vee Gxy)$	3, 4–17 $\exists E$
19	$\sim (\forall x)(\forall y)(Fxy \vee Gxy)$	2, 3–18 $\exists E$
20	$(\forall x)(\forall y)(Fxy \vee Gxy)$	1 R

10.3E

1. Derivability

a. Derive: $(\exists y)(\sim Fy \vee \sim Gy)$

1	$\sim (\forall y)(Fy \& Gy)$	Assumption
2	$(\exists y) \sim (Fy \& Gy)$	1 QN
3	$(\exists y)(\sim Fy \vee \sim Gy)$	2 DeM

c. Derive: $(\exists z)(Az \& \sim Cz)$

1	$(\exists z)(Gz \& Az)$	Assumption			
2	$(\forall y)(Cy \supset \sim Gy)$	Assumption			
3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="padding-right: 10px;">3</td> <td style="border-left: 1px solid black; padding-left: 10px;">$Gh \& Ah$</td> <td style="padding-left: 20px;">A / $\exists E$</td> </tr> </table>	3	$Gh \& Ah$	A / $\exists E$	
3	$Gh \& Ah$	A / $\exists E$			
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="padding-right: 10px;">4</td> <td style="border-left: 1px solid black; padding-left: 10px;">$Ch \supset \sim Gh$</td> <td style="padding-left: 20px;">2 $\forall E$</td> </tr> </table>	4	$Ch \supset \sim Gh$	2 $\forall E$	
4	$Ch \supset \sim Gh$	2 $\forall E$			
5	Gh	3 &E			
6	$\sim \sim Gh$	5 DN			
7	$\sim Ch$	4, 6 MT			
8	Ah	3 &E			
9	$Ah \& \sim Ch$	7, 8 &I			
10	$(\exists z)(Az \& \sim Cz)$	9 $\exists I$			
11	$(\exists z)(Az \& \sim Cz)$	1, 3–10 $\exists E$			

e. Derive: $(\exists x)Cxb$

1	$(\forall x)[(\sim Cxb \vee Hx) \supset Lxx]$	Assumption			
2	$(\exists y) \sim Lyy$	Assumption			
3	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="padding-right: 10px;">3</td> <td style="border-left: 1px solid black; padding-left: 10px;">$\sim Lmm$</td> <td style="padding-left: 20px;">A / $\exists E$</td> </tr> </table>	3	$\sim Lmm$	A / $\exists E$	
3	$\sim Lmm$	A / $\exists E$			
4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="padding-right: 10px;">4</td> <td style="border-left: 1px solid black; padding-left: 10px;">$(\sim Cmb \vee Hm) \supset Lmm$</td> <td style="padding-left: 20px;">1 $\forall E$</td> </tr> </table>	4	$(\sim Cmb \vee Hm) \supset Lmm$	1 $\forall E$	
4	$(\sim Cmb \vee Hm) \supset Lmm$	1 $\forall E$			
5	$\sim (\sim Cmb \vee Hm)$	3, 4 MT			
6	$\sim \sim Cmb \& \sim Hm$	5 DeM			
7	$\sim \sim Cmb$	6 &E			
8	Cmb	7 DN			
9	$(\exists x)Cxb$	8 $\exists I$			
10	$(\exists x)Cxb$	2, 3–9 $\exists E$			

2. Validity

a. Derive: $(\forall y) \sim (Hby \vee Ryy)$

1	$(\forall y) \sim Jx$	Assumption
2	$(\exists y)(Hby \vee Ryy) \supset (\exists x)Jx$	Assumption
3	$\sim (\exists x)Jx$	1 QN
4	$\sim (\exists y)(Hby \vee Ryy)$	2, 3 MT
5	$(\forall y) \sim (Hby \vee Ryy)$	4 QN

c. Derive: $(\forall x)(\forall y)Hxy \ \& \ (\forall x) \sim Tx$

1	$(\forall x) \sim ((\forall y)Hxy \vee Tx)$	Assumption
2	$\sim (\exists y)(Ty \vee (\exists x) \sim Hxy)$	Assumption
3	$(\forall y) \sim (Ty \vee (\exists x) \sim Hxy)$	2 QN
4	$\sim (Ta \vee (\exists x) \sim Hxa)$	3 $\forall E$
5	$\sim Ta \ \& \ \sim (\exists x) \sim Hxa$	4 DeM
6	$\sim (\exists x) \sim Hxa$	5 $\&E$
7	$(\forall x) \sim \sim Hxa$	6 QN
8	$\sim \sim Hba$	7 $\forall E$
9	Hba	8 DN
10	$(\forall y)Hby$	9 $\forall I$
11	$(\forall x)(\forall y)Hxy$	10 $\forall I$
12	$\sim Ta$	5 $\&E$
13	$(\forall x) \sim Tx$	12 $\forall I$
14	$(\forall x)(\forall y)Hxy \ \& \ (\forall x) \sim Tx$	11, 13 $\&I$

e. Derive: $(\exists x) \sim Kxx$

1	$(\forall z)[Kzz \supset (Mz \ \& \ Nz)]$	Assumption
2	$(\exists z) \sim Nz$	Assumption
3	$\sim Ng$	A / $\exists E$
4	$Kgg \supset (Mg \ \& \ Ng)$	1 $\forall E$
5	$\sim Mg \vee \sim Ng$	3 $\vee I$
6	$\sim (Mg \ \& \ Ng)$	5 DeM
7	$\sim Kgg$	4, 6 MT
8	$(\exists x) \sim Kxx$	7 $\exists I$
9	$(\exists x) \sim Kxx$	2, 3–8 $\exists E$

g. Derive: $(\exists w)(Qw \ \& \ Bw) \supset (\forall y)(Lyy \supset \sim Ay)$

1	$(\exists z)Qz \supset (\forall w)(Lww \supset \sim Hw)$	Assumption
2	$(\exists x)Bx \supset (\forall y)(Ay \supset Hy)$	Assumption
3	$(\exists w)(Qw \ \& \ Bw)$	A / $\supset I$
4	$Qm \ \& \ Bm$	A / $\exists E$
5	Qm	4 $\&E$
6	$(\exists z)Qz$	5 $\exists I$
7	$(\forall w)(Lww \supset \sim Hw)$	1, 6 $\supset E$
8	$Lcc \supset \sim Hc$	7 $\forall E$
9	Bm	4 $\&E$
10	$(\exists x)Bx$	9 $\exists I$
11	$(\forall y)(Ay \supset Hy)$	2, 10 $\supset E$
12	$Ac \supset Hc$	11 $\forall E$
13	$\sim Hc \supset \sim Ac$	12 Trans
14	$Lcc \supset \sim Ac$	8, 13 HS
15	$(\forall y)(Lyy \supset \sim Ay)$	14 $\forall I$
16	$(\forall y)(Lyy \supset \sim Ay)$	3, 4–15 $\exists E$
17	$(\exists w)(Qw \ \& \ Bw) \supset (\forall y)(Lyy \supset \sim Ay)$	3–16 $\supset I$

i. Derive: $\sim (\forall x)(\forall y)Bxy \supset (\forall x)(\sim Gx \vee \sim Hx)$

1	$\sim (\forall x)(\sim Gx \vee \sim Hx) \supset (\forall x)[Cx \ \& \ (\forall y)(Ly \supset Axy)]$	Assumption
2	$(\exists x) [Hx \ \& \ (\forall y)(Ly \supset Axy)] \supset (\forall x)(Fx \ \& \ (\forall y)Bxy)$	Assumption
3	$\sim (\forall x)(\sim Gx \vee \sim Hx)$	A / \supset I
4	$(\exists x) \sim (\sim Gx \vee \sim Hx)$	3 QN
5	$\sim (\sim Gi \vee \sim Hi)$	A / \exists I
6	$\sim \sim Gi \ \& \ \sim \sim Hi$	5 DeM
7	$\sim \sim Hi$	6 &E
8	Hi	7 DN
9	$(\forall x)[Cx \ \& \ (\forall y)(Ly \supset Axy)]$	1, 3 \supset E
10	Ci & $(\forall y)(Ly \supset Aiy)$	9 \forall E
11	$(\forall y)(Ly \supset Aiy)$	10 &E
12	Hi & $(\forall y)(Ly \supset Aiy)$	8, 11 &I
13	$(\exists x)[Hx \ \& \ (\forall y)(Ly \supset Axy)]$	12 \exists I
14	$(\forall x)(Fx \ \& \ (\forall y)Bxy)$	2, 13 \supset E
15	Fj & $(\forall y)Bjy$	14 \forall E
16	$(\forall y)Bjy$	15 &E
17	$(\forall x)(\forall y)Bxy$	16 \forall I
18	$(\forall x)(\forall y)Bxy$	4, 5–17 \exists E
19	$\sim (\forall x)(\sim Gx \vee \sim Hx) \supset (\forall x)(\forall y)Bxy$	3–18 \supset I
20	$\sim (\forall x)(\forall y)Bxy \supset \sim \sim (\forall x)(\sim Gx \vee \sim Hx)$	19 Trans
21	$\sim (\forall x)(\forall y)Bxy \supset (\forall x)(\sim Gx \vee \sim Hx)$	20 DN

3. Theorems

a. Derive: $(\forall x)(Ax \supset Bx) \supset (\forall x)(Bx \vee \sim Ax)$

1	$(\forall x)(Ax \supset Bx)$	A / \supset I
2	$(\forall x)(\sim Ax \vee Bx)$	1 Impl
3	$(\forall x)(Bx \vee \sim Ax)$	2 Com
4	$(\forall x)(Ax \supset Bx) \supset (\forall x)(Bx \vee \sim Ax)$	1–3 \supset I

c. Derive: $\sim (\exists x)(Ax \vee Bx) \supset (\forall x) \sim Ax$

1	$\sim (\exists x)(Ax \vee Bx)$	A / \supset I
2	$(\forall x) \sim (Ax \vee Bx)$	1 QN
3	$\sim (Ac \vee Bc)$	2 \forall E
4	$\sim Ac \ \& \ \sim Bc$	3 DeM
5	$\sim Ac$	4 &E
6	$(\forall x) \sim Ax$	5 \forall I
7	$\sim (\exists x)(Ax \vee Bx) \supset (\forall x) \sim Ax$	1–6 \supset I

e. Derive: $((\exists x)Ax \supset (\exists x)Bx) \supset (\exists x)(Ax \supset Bx)$

1	$\sim (\exists x)(Ax \supset Bx)$		A / \supset I
2	$(\forall x) \sim (Ax \supset Bx)$		1 QN
3	$\sim (Ac \supset Bc)$		2 \forall E
4	$\sim (\sim Ac \vee Bc)$		3 Impl
5	$\sim \sim Ac \ \& \ \sim Bc$		4 DeM
6	$\sim \sim Ac$		5 &E
7	$(\exists x) \sim \sim Ax$		6 \exists I
8	$\sim (\forall x) \sim Ax$		7 QN
9	$\sim \sim (\exists x)Ax$		8 QN
10	$\sim Bc$		5 &E
11	$(\forall x) \sim Bx$		10 \forall I
12	$\sim (\exists x)Bx$		11 QN
13	$\sim \sim (\exists x)Ax \ \& \ \sim (\exists x)Bx$		9, 12 &I
14	$\sim (\sim (\exists x)Ax \vee (\exists x)Bx)$		13 DeM
15	$\sim ((\exists x)Ax \supset (\exists x)Bx)$		14 Impl
16	$\sim (\exists x)(Ax \supset Bx) \supset \sim ((\exists x)Ax \supset (\exists x)Bx)$		1–15 \supset I
17	$((\exists x)Ax \supset (\exists x)Bx) \supset (\exists x)(Ax \supset Bx)$		16 Trans

4. Equivalence

a. Derive: $(\exists x)(Ax \ \& \ \sim Bx)$

1	$\sim (\forall x)(Ax \supset Bx)$		Assumption
2	$(\exists x) \sim (Ax \supset Bx)$		1 QN
3	$(\exists x) \sim (\sim Ax \vee Bx)$		2 Impl
4	$(\exists x)(\sim \sim Ax \ \& \ \sim Bx)$		3 DeM
5	$(\exists x)(Ax \ \& \ \sim Bx)$		4 DN

Derive: $\sim (\forall x)(Ax \supset Bx)$

1	$(\exists x)(Ax \ \& \ \sim Bx)$		Assumption
2	$(\exists x)(\sim \sim Ax \ \& \ \sim Bx)$		1 DN
3	$(\exists x) \sim (\sim Ax \vee Bx)$		2 DeM
4	$(\exists x) \sim (Ax \supset Bx)$		3 Impl
5	$\sim (\forall x)(Ax \supset Bx)$		4 QN

c. Derive: $(\exists x)[\sim Ax \vee (\sim Cx \supset \sim Bx)]$

1	$\sim (\forall x) \sim [(Ax \ \& \ Bx) \supset Cx]$		Assumption
2	$(\exists x) \sim \sim [(Ax \ \& \ Bx) \supset Cx]$		1 QN
3	$(\exists x)[(Ax \ \& \ Bx) \supset Cx]$		2 DN
4	$(\exists x)[Ax \supset (Bx \supset Cx)]$		3 Exp
5	$(\exists x)[\sim Ax \vee (Bx \supset Cx)]$		4 Impl
6	$(\exists x)[\sim Ax \vee (\sim Cx \supset \sim Bx)]$		5 Trans

Derive: $\sim (\forall x) \sim [(Ax \ \& \ Bx) \supset Cx]$

1	$(\exists x)[\sim Ax \vee (\sim Cx \supset \sim Bx)]$	Assumption
2	$(\exists x)[\sim Ax \vee (Bx \supset Cx)]$	1 Trans
3	$(\exists x)[Ax \supset (Bx \supset Cx)]$	2 Impl
4	$(\exists x)[(Ax \ \& \ Bx) \supset Cx]$	3 Exp
5	$\sim \sim (\exists x)[(Ax \ \& \ Bx) \supset Cx]$	4 DN
6	$\sim (\forall x) \sim [(Ax \ \& \ Bx) \supset Cx]$	5 QN

e. Derive: $\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ (Ax \vee Bx)]$

1	$(\forall x)(Ax \equiv Bx)$	Assumption
2	$\sim \sim (\forall x)(Ax \equiv Bx)$	1 DN
3	$\sim (\exists x) \sim (Ax \equiv Bx)$	2 QN
4	$\sim (\exists x) \sim [(Ax \ \& \ Bx) \vee (\sim Ax \ \& \ \sim Bx)]$	3 Equiv
5	$\sim (\exists x)[\sim (Ax \ \& \ Bx) \ \& \ \sim (\sim Ax \ \& \ \sim Bx)]$	4 DeM
6	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ \sim (\sim Ax \ \& \ \sim Bx)]$	5 DeM
7	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ (\sim \sim Ax \vee \sim \sim Bx)]$	6 DeM
8	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ (Ax \vee \sim \sim Bx)]$	7 DN
9	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ (Ax \vee Bx)]$	8 DN

Derive: $(\forall x)(Ax \equiv Bx)$

1	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ (Ax \vee Bx)]$	Assumption
2	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ (Ax \vee \sim \sim Bx)]$	1 DN
3	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ (\sim \sim Ax \vee \sim \sim Bx)]$	2 DN
4	$\sim (\exists x)[(\sim Ax \vee \sim Bx) \ \& \ \sim (\sim Ax \ \& \ \sim Bx)]$	3 DeM
5	$\sim (\exists x)[\sim (Ax \ \& \ Bx) \ \& \ \sim (\sim Ax \ \& \ \sim Bx)]$	4 DeM
6	$\sim (\exists x) \sim [(Ax \ \& \ Bx) \vee (\sim Ax \ \& \ \sim Bx)]$	5 DeM
7	$\sim (\exists x) \sim (Ax \equiv Bx)$	6 Equiv
8	$\sim \sim (\forall x)(Ax \equiv Bx)$	7 QN
9	$(\forall x)(Ax \equiv Bx)$	8 DN

5. Inconsistency

a. Derive: $Jc, \sim Jc$

1	$[(\forall x)(Mx \equiv Jx) \ \& \ \sim Mc] \ \& \ (\forall x)Jx$	Assumption
2	$(\forall x)(Mx \equiv Jx) \ \& \ \sim Mc$	1 &E
3	$(\forall x)(Mx \equiv Jx)$	2 &E
4	$Mc \equiv Jc$	3 $\forall E$
5	$(Mc \supset Jc) \ \& \ (Jc \supset Mc)$	4 Equiv
6	$Jc \supset Mc$	5 &E
7	$\sim Mc$	2 &E
8	$\sim Jc$	6, 7 MT
9	$(\forall x)Jx$	1 &E
10	Jc	9 $\forall E$

c. Derive: $(\exists w)Cww, \sim (\exists w)Cww$

1	$(\forall x)(\forall y)Lxy \supset \sim (\exists z)Tz$	Assumption
2	$(\forall x)(\forall y)Lxy \supset ((\exists w)Cww \vee (\exists z)Tz)$	Assumption
3	$(\sim (\forall x)(\forall y)Lxy \vee (\forall z)Bzzk) \ \&$ $(\sim (\forall z)Bzzk \vee \sim (\exists w)Cww)$	Assumption
4	$(\forall x)(\forall y)Lxy$	Assumption
5	$\sim (\exists z)Tz$	1, 4 $\supset E$
6	$(\exists w)Cww \vee (\exists z)Tz$	2, 4 $\supset E$
7	$(\exists w)Cww$	5, 6 DS
8	$\sim (\forall x)(\forall y)Lxy \vee (\forall z)Bzzk$	3 &E
9	$(\forall x)(\forall y)Lxy \supset (\forall z)Bzzk$	8 Impl
10	$(\forall z)Bzzk$	4, 9 $\supset E$
11	$\sim (\forall z)Bzzk \vee \sim (\exists w)Cww$	3 &E
12	$(\forall z)Bzzk \supset \sim (\exists w)Cww$	11 Impl
13	$\sim (\exists w)Cww$	10, 12 $\supset E$

e. Derive: $Hc, \sim Hc$

1	$(\forall x)(\forall y)(Gxy \supset Hc)$	Assumption
2	$(\exists x)Gix \ \& \ (\forall x)(\forall y)(\forall z)Lxyz$	Assumption
3	$\sim Lcib \vee \sim (Hc \vee Hc)$	Assumption
4	$(\exists x)Gix$	2 &E
5	Gik	A / $\exists E$
6	$(\forall y)(Giy \supset Hc)$	1 $\forall E$
7	$Gik \supset Hc$	6 $\forall E$
8	Hc	5, 7 $\supset E$
9	Hc	4, 5–8 $\exists E$
10	$(\forall x)(\forall y)(\forall z)Lxyz$	2 &E
11	$(\forall y)(\forall z)Lcyz$	10 $\forall E$
12	$(\forall z)Lciz$	11 $\forall E$
13	$Lcib$	12 $\forall E$
14	$\sim \sim Lcib$	13 DN
15	$\sim (Hc \vee Hc)$	3, 14 DS
16	$\sim Hc$	15 Idem

6. a. Suppose there is a sentence on an accessible line **i** of a derivation to which Universal Elimination can be properly applied at line **n**. The sentence that would be derived by Universal Elimination can also be derived by using the routine beginning at line **n**:

i	$(\forall \mathbf{x})\mathbf{P}$	
n	$\sim \mathbf{P}(\mathbf{a}/\mathbf{x})$	A / $\sim E$
n + 1	$(\exists \mathbf{x}) \sim \mathbf{P}$	n $\exists I$
n + 2	$\sim (\forall \mathbf{x})\mathbf{P}$	n + 1 QN
n + 3	$(\forall \mathbf{x})\mathbf{P}$	i R
n + 4	$\mathbf{P}(\mathbf{a}/\mathbf{x})$	n - n + 3 $\sim E$

Suppose there is a sentence on an accessible line i of a derivation to which Universal Introduction can be properly applied at line n . The sentence that would be derived by Universal Introduction can also be derived by using the routine beginning at line n :

i n $n + 1$ $n + 2$ $n + 3$ $n + 4$ $n + 5$ $n + 6$ $n + 7$ $n + 8$ $n + 9$	$P(a/x)$ $\sim (\forall x)P$ $(\exists x) \sim P$ $\sim P(a/x)$ $\sim (\forall x)P$ $P(a/x)$ $\sim P(a/x)$ $(\forall x)P$ $(\forall x)P$ $\sim (\forall x)P$ $(\forall x)P$	$A / \sim E$ $n \text{ QN}$ $A / \exists E$ $A / \sim E$ $i \text{ R}$ $n + 2 \text{ R}$ $n + 3 - n + 5 \sim E$ $n + 1, n + 2 - n + 6 \exists E$ $n \text{ R}$ $n - n + 8 \sim E$
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No restriction on the use of Existential Elimination was violated at line $n + 7$. We assumed that we could have applied Universal Introduction at line n to $P(a/x)$ on line i . So a does not occur in any undischarged assumption prior to line n , and a does not occur in $(\forall x)P$. So a does not occur in P . Hence

- (i) a does not occur in any undischarged assumption prior to $n + 7$. Note that the assumptions on lines $n + 2$ and $n + 3$ have been discharged and that a cannot occur in the assumption on line n , for a does not occur in P .
- (ii) a does not occur in $(\exists x) \sim P$, for a does not occur in P .
- (iii) a does not occur in $(\forall x)P$, for a does not occur in P .

10.4E EXERCISES

1. Theorems

a. Derive: $a = b \supset b = a$

1 2 3 4	$a = b$ $a = a$ $b = a$ $a = b \supset b = a$	$A / \supset I$ $1, 1 =E$ $1, 2 =E$ $1-3 \supset I$
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c. Derive: $(\sim a = b \ \& \ b = c) \supset \sim a = c$

1 2 3 4 5	$\sim a = b \ \& \ b = c$ $\sim a = b$ $b = c$ $\sim a = c$ $(\sim a = b \ \& \ b = c) \supset \sim a = c$	$A / \supset I$ $1 \ \&E$ $1 \ \&E$ $2, 3 =E$ $1-4 \supset I$
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e. Derive: $\sim a = c \supset (\sim a = b \vee \sim b = c)$

1	$\sim a = c$		A / \supset I
2	$\sim (\sim a = b \vee \sim b = c)$		A / \sim E
3	$\sim a = b$		A / \sim E
4	$\sim a = b \vee \sim b = c$		3 \vee I
5	$\sim (\sim a = b \vee \sim b = c)$		2 R
6	$a = b$		3-5 \sim E
7	$\sim b = c$		1, 6 =E
8	$\sim a = b \vee \sim b = c$		7 \vee I
9	$\sim (\sim a = b \vee \sim b = c)$		2 R
10	$\sim a = b \vee \sim b = c$		2-9 \sim E
11	$\sim a = c \supset (\sim a = b \vee \sim b = c)$		1-10 \supset I

2. Validity

a. Derive: $\sim (\forall x)Bxx$

1	$a = b \ \& \ \sim Bab$		Assumption
2	$\sim Bab$		1 &E
3	$a = b$		1 &E
4	$(\forall x)Bxx$		A / \sim I
5	Baa		4 \forall E
6	$\sim Baa$		2, 3 =E
7	$\sim (\forall x)Bxx$		4-6 \sim I

c. Derive: Hii

1	$(\forall z)[Gz \supset (\forall y)(Ky \supset Hzy)]$		Assumption
2	$(Ki \ \& \ Gj) \ \& \ i = j$		Assumption
3	$Gj \supset (\forall y)(Ky \supset Hjy)$		1 \forall E
4	$Ki \ \& \ Gj$		2 &E
5	Gj		4 &E
6	$(\forall y)(Ky \supset Hjy)$		3, 5 \supset E
7	$Ki \supset Hji$		7 \forall E
8	Ki		4 &E
9	Hji		7, 8 \supset E
10	$i = j$		2 &E
11	Hii		9, 10 =E

e. Derive: $Ka \vee \sim Kb$

1	a = b		
2	$\sim (Ka \vee \sim Ka)$		A / \sim E
3	Ka		A / \sim I
4	$Ka \vee \sim Ka$		3 \vee I
5	$\sim (Ka \vee \sim Ka)$		2 R
6	$\sim Ka$		3-5 \sim I
7	$Ka \vee \sim Ka$		6 \vee I
8	$\sim (Ka \vee \sim Ka)$		2 R
9	$Ka \vee \sim Ka$		2-8 \sim E
10	$Ka \vee \sim Kb$		1, 9 =E

3. Theorems

a. Derive: $(\forall x)(x = x \vee \sim x = x)$

1	$(\forall x)x = x$		
2	a = a		=I
3	$a = a \vee \sim a = a$		1 \forall E
4	$(\forall x)(x = x \vee \sim x = x)$		2 \vee I
			3 \forall I

c. Derive: $(\forall x)(\forall y)(x = y \equiv y = x)$

1	a = b		
2	a = a		A / \equiv I
3	b = a		1, 1 =E
4	b = a		1, 2 =E
5	b = b		A / \equiv I
6	a = b		4, 4 =E
7	a = b \equiv b = a		4, 5 =E
8	$(\forall y)(a = y \equiv y = a)$		1-3, 4-6 \equiv I
9	$(\forall x)(\forall y)(x = y \equiv y = x)$		7 \forall I
			8 \forall I

e. Derive: $\sim (\exists x) \sim x = x$

1	$(\exists x) \sim x = x$		
2	$\sim a = a$		A / \sim I
3	$(\exists x) \sim x = x$		A / \exists E
4	$(\forall x)x = x$		A / \sim I
5	a = a		=I
6	$\sim a = a$		4 \forall E
7	$\sim (\exists x) \sim x = x$		2 R
8	$\sim (\exists x) \sim x = x$		3-6, \sim I
9	$(\exists x) \sim x = x$		1, 2-7 \exists E
10	$\sim (\exists x) \sim x = x$		1 R
			1-9 \sim I

4. Validity

a. Derive: $(\exists x)(\exists y)[(Ex \ \& \ Ey) \ \& \ \sim x = y]$

1	$\sim t = f$	Assumption
2	$Et \ \& \ Ef$	Assumption
3	$(Et \ \& \ Ef) \ \& \ \sim t = f$	1, 2 &I
4	$(\exists y)[(Et \ \& \ Ey) \ \& \ \sim t = y]$	3 \exists I
5	$(\exists x)(\exists y)[(Ex \ \& \ Ey) \ \& \ \sim x = y]$	4 \exists I

c. Derive: $\sim s = b$

1	$\sim Ass \ \& \ Aqb$	Assumption
2	$(\forall x)[(\exists y)Ayx \supset Abx]$	Assumption
3	$s = b$	A / \sim I
4	$(\exists y)Ayb \supset Abb$	2 \forall E
5	Aqb	1 &E
6	$(\exists y)Ayb$	5 \exists I
7	Abb	4, 6 \supset E
8	$\sim Ass$	1 &E
9	$\sim Abb$	3, 8 =E
10	$\sim s = b$	3-9 \sim I

e. Derive: $(\exists x)[(Rxe \ \& \ Pxa) \ \& \ (\sim x = e \ \& \ \sim x = a)]$

1	$(\exists x)(Rxe \ \& \ Pxa)$	Assumption
2	$\sim Ree$	Assumption
3	$\sim Paa$	Assumption
4	$Rie \ \& \ Pia$	A / \exists E
5	$i = e$	A / \sim I
6	Rie	4 &E
7	Ree	5, 6 =E
8	$\sim Ree$	2 R
9	$\sim i = e$	5-8 \sim I
10	$i = a$	A / \sim I
11	Pia	4 &E
12	Paa	10, 11 =E
13	$\sim Paa$	3 R
14	$\sim i = a$	10-13 \sim I
15	$\sim i = e \ \& \ \sim i = a$	9, 14 &I
16	$(Rie \ \& \ Pia) \ \& \ (\sim i = e \ \& \ \sim i = a)$	4, 15 &I
17	$(\exists x)[(Rxe \ \& \ Pxa) \ \& \ (\sim x = e \ \& \ \sim x = a)]$	16 \exists I
18	$(\exists x)[(Rxe \ \& \ Pxa) \ \& \ (\sim x = e \ \& \ \sim x = a)]$	1, 4-17 \exists E

5.a.	1	$(\exists x)Sx$	Assumption
	2	$Sg(f)$	A / $\exists E$
	3	$(\exists x)Sg(x)$	2 $\exists I$
	4	$(\exists x)Sg(x)$	1, 2-3 $\exists E$

Line 2 is a mistake as an instantiating individual constant must be used, *not* a closed complex term.

c. Correctly done.

e.	1	$(\forall x)Lxxx$	Assumption
	2	$Lf(a,a)a$	1 $\forall E$
	3	$(\forall x)Lf(x,x)x$	2 $\forall I$

Line 2 is a mistake. Universal Elimination does not permit using both a closed complex term and at the same time an individual constant in the substitution instance, not to mention that all three occurrences of the variable 'x' must be replaced.

g.	1	$(\forall x)Rf(x,x)$	Assumption
	2	$Rf(c,c)$	1 $\forall E$
	3	$(\forall y)Ry$	2 $\forall I$

Line 3 is a mistake. Universal Introduction cannot be applied using a closed complex term.

i. Correctly done.

6. Theorems in PDE:

a. Derive: $(\forall x)(\exists y)f(x) = y$

1	$(\forall x)x = x$	$=I$
2	$f(a) = f(a)$	1 $\forall E$
3	$(\exists y)f(a) = y$	2 $\exists I$
4	$(\forall x)(\exists y)f(x) = y$	3 $\forall I$

c. Derive: $(\forall x)Ff(x) \supset (\forall x)Ff(g(x))$

1	$(\forall x)Ff(x)$	A / $\supset I$
2	$Ff(g(a))$	1 $\forall E$
3	$(\forall x)Ff(g(x))$	2 $\forall I$
4	$(\forall x)Ff(x) \supset (\forall x)Ff(g(x))$	1-3 $\supset I$

e. Derive: $(\forall x)(f(f(x)) = x \supset f(f(f(f(x)))) = x)$

1		$f(f(a)) = a$	A / \supset I
2		$f(f(f(f(a)))) = a$	1, 1 =E
3		$f(f(a)) = a \supset f(f(f(f(a)))) = a$	1-2 \supset I
4		$(\forall x)(f(f(x)) = x \supset f(f(f(f(x)))) = x)$	3 \forall I

g. Derive: $(\forall x)(\forall y)[(f(x) = y \ \& \ f(y) = x) \supset x = f(f(x))]$

1		$f(a) = b \ \& \ f(b) = a$	A / \supset I
2		$f(b) = a$	1 &E
3		$f(b) = f(b)$	2, 2 =E
4		$a = f(b)$	2, 3 =E
5		$f(a) = b$	1 &E
6		$a = f(f(a))$	4, 5 =E
7		$(f(a) = b \ \& \ f(b) = a) \supset a = f(f(a))$	1-6 \supset I
8		$(\forall y)[(f(a) = y \ \& \ f(y) = a) \supset a = f(f(a))]$	7 \forall I
9		$(\forall x)(\forall y)[(f(x) = y \ \& \ f(y) = x) \supset x = f(f(x))]$	8 \forall I

7. Validity in PDE:

a. Derive: $(\forall x)Gf(x)f(f(x))$

1		$(\forall x)(Bx \supset Gxf(x))$	Assumption
2		$(\forall x)Bf(x)$	Assumption
3		$Bf(a) \supset Gf(a)f(f(a))$	1 \forall E
4		$Bf(a)$	2 \forall E
5		$Gf(a)f(f(a))$	3, 4 \supset E
6		$(\forall x)Gf(x)f(f(x))$	5 \forall I

c. Derive: $\sim f(a) = b$

1		$(\forall x)(\forall y)(f(x) = y \supset Myxc)$	Assumption		
2		$\sim Mbac \ \& \ \sim Mabac$	Assumption		
3		$(\forall y)(f(a) = y \supset Myac)$	1 \forall E		
4		$f(a) = b \supset Mbac$	3 \forall E		
5			$f(a) = b$	A / \sim I	
6				$Mbac$	4, 5 \supset E
7				$\sim Mbac$	2 &E
8				$\sim f(a) = b$	5-7 \sim I

e. Derive: $(\exists x)Lxf(x)g(x)$

1		$(\exists x)(\forall y)(\forall z)Lxyz$	Assumption
2		$(\forall y)(\forall z)Layz$	A / $\exists E$
3		$(\forall z)Laf(a)z$	2 $\forall E$
4		$Laf(a)g(a)$	3 $\forall E$
5		$(\exists x)Lxf(x)g(x)$	4 $\exists I$
6		$(\exists x)Lxf(x)g(x)$	1, 2-5 $\exists E$

g. Derive: $(\forall x)Df(x)f(x)$

1		$(\forall x)[Zx \supset (\forall y)(\sim Dxy \equiv Hf(f(y)))]$	Assumption
2		$(\forall x)(Zx \ \& \ \sim Hx)$	Assumption
3		$Zf(a) \supset (\forall y)(\sim Df(a)y \equiv Hf(f(y)))$	1 $\forall E$
4		$Zf(a) \ \& \ \sim Hf(a)$	2 $\forall E$
5		$Zf(a)$	4 $\&E$
6		$(\forall y)(\sim Df(a)y \equiv Hf(f(y)))$	3, 5 $\supset E$
7		$\sim Df(a)f(a) \equiv Hf(f(a))$	6 $\forall E$
8		$\sim Df(a)f(a)$	A / $\sim E$
9		$Hf(f(a))$	7, 8 $\equiv E$
10		$Zf(f(a)) \ \& \ \sim Hf(f(a))$	2 $\forall E$
11		$\sim Hf(f(a))$	10 $\&E$
12		$Df(a)f(a)$	8-11 $\sim E$
13		$(\forall x)Df(x)f(x)$	12 $\forall I$